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by Pere Ara and Martin Mathieu
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When is the second local multiplier algebra of a $C^*$-algebra equal to the first?

Pere Ara and Martin Mathieu

Abstract

We discuss necessary as well as sufficient conditions for the second iterated local multiplier algebra of a separable $C^*$-algebra to agree with the first.

1. Introduction

After the first example of a $C^*$-algebra $A$ with the property that the second local multiplier algebra $M_{\text{loc}}(M_{\text{loc}}(A))$ of $A$ differs from its first, $M_{\text{loc}}(A)$, was found in [3]—thus answering a question first raised in [17]—, the behaviour of higher local multiplier algebras began to attract some attention; see, e.g., [4], [7], [8]. That the local multiplier algebra can have a somewhat complicated structure was already exhibited in [1], where an example of a non-simple unital $C^*$-algebra $A$ was given such that $M_{\text{loc}}(A)$ is simple (and hence, evidently, $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$ in this case).

It was proved in [21] that, if $A$ is a separable unital $C^*$-algebra, $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$, provided the primitive ideal space Prim($A$) contains a dense $G_\delta$ subset of closed points. One of our goals here is to see how this result can be obtained in a straightforward manner using the techniques developed in [5]. The key to our argument is the following observation. Every element in $M_{\text{loc}}(A)$ can be realised as a bounded continuous section, defined on a dense $G_\delta$ subset of Prim($A$), with values in the upper semicontinuous $C^*$-bundle canonically associated to the multiplier sheaf of $A$. The second local multiplier algebra $M_{\text{loc}}(M_{\text{loc}}(A))$ is contained in the injective envelope $I(A)$ of $A$, cf. [12], [4], and every element of $I(A)$ has a similar description as a continuous section of a $C^*$-bundle corresponding to the injective envelope sheaf of $A$. To show that $M_{\text{loc}}(M_{\text{loc}}(A)) \subseteq M_{\text{loc}}(A)$ it thus suffices to relate sections of these bundles in an appropriate way. In fact, we shall obtain a more general result in Section 4 which, in particular, unifies the commutative and the unital case. The notion of a quasicentral $C^*$-algebra, first studied by Delaroche [9], [10], turns out to be crucial.

It emerges, however, that the short answer to the long question in this paper’s title is: rarely. In Section 3, we provide a systematic approach to producing separable $C^*$-algebras with the property that their second local multiplier algebra contains the first as a proper $C^*$-subalgebra. We obtain a quick proof of Somerset’s result [21] that $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}^{(3)}(A)$ for a separable $C^*$-algebra $A$ which has a dense $G_\delta$ subset of closed points in its primitive spectrum in Theorem 3.2 below. In our approach, the injective envelope is employed as a ‘universe’ in which all $C^*$-algebras considered are contained as $C^*$-subalgebras. However, in contrast to previous studies, we do not need additional information on the injective envelope itself.

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In the following we will focus on separable $C^*$-algebras for a variety of reasons. One of them is the non-commutative Tietze extension theorem, another one the need for a strictly positive element in the bounded central closure of the $C^*$-algebra. Moreover, just as in Somerset’s paper [21], Polish spaces (in the primitive spectrum) will play a decisive role. Sections 2 and 3 are fairly self-contained, while Section 4 relies on the sheaf theory developed in [5].

2. Preliminaries

For a $C^*$-algebra $A$, we denote by $\text{Prim}(A)$ its primitive ideal space (with the Jacobson topology); this is second countable if $A$ is separable. For an open subset $U \subseteq \text{Prim}(A)$, let $A(U)$ stand for the closed ideal of $A$ corresponding to $U$. (Hence, $t \in U$ if and only if $A(U) \not\supseteq t$.) We denote by $\mathcal{D}$ and $\mathcal{T}$ the sets of dense open subsets and dense $G_δ$ subsets of $\text{Prim}(A)$, respectively, and consider them directed under reverse inclusion. The local multiplier $M_{\text{loc}}(A)$ is defined by $M_{\text{loc}}(A) = \lim_{U \in \mathcal{D}} M(A(U))$, where, for $U, V \in \mathcal{D}$ with $V \subseteq U$, the injective *-homomorphism $M(A(U)) \to M(A(V))$ is given by restriction. We put $Z = Z(M_{\text{loc}}(A))$, the centre of $M_{\text{loc}}(A)$. For more details on, and properties of, $M_{\text{loc}}(A)$, we refer to [2].

A point $t \in \text{Prim}(A)$ is said to be separated if $t$ and every point $t' \in \text{Prim}(A)$ which is not in the closure of $\{t\}$ can be separated by disjoint neighbourhoods. Let $\text{Sep}(A)$ be the set of all separated points of a $C^*$-algebra $A$. If $A$ is separable then $\text{Sep}(A)$ is a dense $G_δ$ subset of $\text{Prim}(A)$ [11, Théorème 19].

The following result is useful when computing the norm of a (local) multiplier.

**Lemma 2.1.** Let $A$ be a separable $C^*$-algebra, and let $T \subseteq \text{Sep}(A)$ be a dense $G_δ$ subset. For a countable family $\{f_n \mid n \in \mathbb{N}\}$ of bounded lower semicontinuous real-valued functions on $T$ there exists a dense $G_δ$ subset $T' \subseteq T$ such that $f_n|_{T'}$ is continuous for each $n \in \mathbb{N}$.

This is an immediate consequence of the following well-known facts: $\text{Sep}(A)$ is a Polish space (that is, homeomorphic to a separable, complete metric space) by [11, Corollaire 20] and hence any $G_δ$ subset of $\text{Sep}(A)$ is a Polish space [18, 4.2.2]; every Polish space is a Baire space [18, 4.2.5]; any bounded Borel function into $\mathbb{R}$ defined on a Polish space can be restricted to a continuous function on some dense $G_δ$ subset of the domain [15, Sect. 32.I].

In [21], p. 322, Somerset introduces an interesting $C^*$-subalgebra of $M_{\text{loc}}(A)$, which we will denote by $K_A$. $K_A$ is the closure of the set of all elements of the form $\sum_{n\in\mathbb{N}} a_n z_n$, where $\{a_n\} \subseteq A$ is a bounded family and $\{z_n\} \subseteq Z$ consists of mutually orthogonal projections. (These infinite sums exist in $M_{\text{loc}}(A)$ by [2, Lemma 3.3.6], for example. Note also that $Z$ is countably decomposable since $A$ is separable.) It is easy to see that, if the above family $\{a_n\}$ is chosen from $K_A$ instead of $A$, then the sum $\sum_{n\in\mathbb{N}} a_n z_n$ still belongs to $K_A$ ([21, Lemma 2.5]).

The significance of the $C^*$-subalgebra $K_A$ is explained by the following result. Let $\mathcal{I}_{\text{ce}}(A)$ denote the set of all closed essential ideals of a $C^*$-algebra $A$. We denote by $M^{(n)}_{\text{loc}}(A) = M_{\text{loc}}(M^{(n-1)}_{\text{loc}}(A))$, $n \geq 2$ the $n$-fold iterated local multiplier algebra of $A$.

**Lemma 2.2.** Let $A$ be a $C^*$-algebra such that $K_A \in \mathcal{I}_{\text{ce}}(M_{\text{loc}}(A))$.

(i) If $K_I = K_A$ for all $I \in \mathcal{I}_{\text{ce}}(A)$ then $M_{\text{loc}}(K_A) = M(K_A)$.

(ii) If $M_{\text{loc}}(K_A) = M(K_A)$ then $M^{(n+1)}_{\text{loc}}(A) = M^{(n)}_{\text{loc}}(A)$ for all $n \geq 2$.

**Proof.** Let $J \in \mathcal{I}_{\text{ce}}(K_A)$; then $M(K_A) \subseteq M(J)$. Let $I = J \cap A$; then $I \in \mathcal{I}_{\text{ce}}(A)$ by [2, Lemma 2.3.2]. By assumption, we therefore have $K_I = K_A$. Let $m \in M(J)$. As $mI \subseteq K_A$,
whenever \( \{x_n\} \) is a bounded family in \( I \) and \( \{z_n\} \) is a family of mutually orthogonal projections in \( Z \), we obtain

\[
m(\sum_n x_nz_n) = \sum_n mx_nz_n \in K_A
\]

entailing that \( mK_A = mK_I \subseteq K_A \), that is, \( m \in M(K_A) \). Consequently, \( M(J) \subseteq M(K_A) \) which implies (i).

Towards (ii) observe that \( M(K_A) = M_{loc}(K_A) = M_{loc}(M_{loc}(A)) \) by hypothesis. Let \( J \in \mathcal{K}_{ce}(M_{loc}(A)) \). Then \( J \cap K_A \in \mathcal{K}_{ce}(M_{loc}(A)) \) and, since \( J \in \mathcal{K}_{ce}(M(K_A)) \), \( J \cap K_A \in \mathcal{K}_{ce}(J) \). As a result,

\[
M(J) \subseteq M(J \cap K_A) \subseteq M_{loc}(M_{loc}(A)) = M(K_A)
\]

and the reverse inclusion \( M(K_A) \subseteq M(J) \) is obvious. We conclude that \( M^{(3)}_{loc}(A) = M(K_A) = M^{(2)}_{loc}(A) \) which entails the result.

The next result tells us how to detect multipliers of \( K_A \) inside \( I(A) \).

**Lemma 2.3.** Let \( A \) be a separable \( C^* \)-algebra and let \( y \in I(A) \). If \( ya \in K_A \) for all \( a \in A \) then \( y \in M(K_A) \).

**Proof.** It suffices to show that \( y \sum_{n=1}^{\infty} z_n a_n = \sum_{n=1}^{\infty} z_n ya_n \) whenever \( \{a_n \mid n \in \mathbb{N}\} \subseteq A \) is a bounded family and \( \{z_n \mid n \in \mathbb{N}\} \subseteq Z \) consists of mutually orthogonal projections, by [21, Lemma 2.5]. Without loss of generality we can assume that \( \sum_{n=1}^{\infty} z_n = 1 \).

Putting \( y' = y \sum_{n=1}^{\infty} z_n a_n \in I(A) \) we observe that

\[
z_j y'_j = y z_j \sum_{n=1}^{\infty} z_n a_n = y z_j a_j = z_j y a_j \in K_A
\]

by hypothesis. It is therefore enough to prove that, if \( y_j \in I(A) \) and \( y_j' z_j \in K_A \) for all \( j \in \mathbb{N} \), where \( \{z_j \mid j \in \mathbb{N}\} \subseteq Z \) consists of mutually orthogonal projections with \( \sum_{j=1}^{\infty} z_j = 1 \), then

\[
y' = \sum_{j=1}^{\infty} y_j' z_j
\]

where the latter is computed in \( K_A \).

The assumption \( y_j' z_j \in K_A \) for all \( j \in \mathbb{N} \) enables us to write \( \sum_{j=1}^{\infty} y_j' z_j = \sum_{i=1}^{\infty} w_i a_i \) for some bounded sequence \( \{a_i \}_{i \in \mathbb{N}} \) in \( A \) and a sequence \( \{w_i \}_{i \in \mathbb{N}} \) consisting of mutually orthogonal central projections with \( \sum_{i=1}^{\infty} w_i = 1 \). For each \( n \in \mathbb{N} \),

\[
(w_1 + \ldots + w_n)y' = \sum_{i=1}^{n} w_i a_i.
\]

Each projection \( w_i \) comes with a closed ideal \( I_i = w_iM_{loc}(A) \cap A \) and the \( C^* \)-direct sum \( I = \bigoplus_{i=1}^{\infty} I_i \) is a closed essential ideal of \( A \). For \( x_i \in I_i \), \( 1 \leq i \leq n \), we have

\[
(y' - \sum_{i=1}^{\infty} w_i a_i)(x_1 + \ldots + x_n) = (y' - \sum_{i=1}^{\infty} w_i a_i)(w_1 + \ldots + w_n)(x_1 + \ldots + x_n)
\]

\[
= (\sum_{i=1}^{n} w_i a_i - \sum_{i=1}^{n} w_i a_i)(x_1 + \ldots + x_n) = 0.
\]

Therefore \( (y' - \sum_{i=1}^{\infty} w_i a_i) x = 0 \) for all \( x \in I \) which implies that \( y' = \sum_{i=1}^{\infty} w_i a_i \) by [4, Proposition 2.12].

Recall that the bounded central closure, \( \mathcal{B}^* \), of a \( C^* \)-algebra \( A \) is the \( C^* \)-subalgebra \( TZ \) of \( M_{loc}(A) \) [2, Section 3.2]. If \( A \) is separable then \( \mathcal{B}^* \) is \( \sigma \)-unital, which will be useful in Section 3.
In Section 4, we shall need the following auxiliary result whose proof is analogous to the one of [21, Lemma 2.2] but we include it here for completeness.

**Lemma 2.4.** Let $A$ be a separable $C^*$-algebra, $B$ a $C^*$-subalgebra of $M_{loc}(A)$ containing $A$, and $J$ a closed essential ideal of $B$. There is $h \in J$ such that $hz \neq 0$ for each non-zero projection $z \in Z$.

**Proof.** By [4, Proposition 2.14], $I(A) = I(B) = I(M_{loc}(A))$ and thus $Z(M_{loc}(B)) = Z$ by [4, Theorem 4.12]. For $x \in M_{loc}(A)$, let $c(x)$ denote the central support of $x$, see [2], page 52 and Remark 3.3.3. Let $\{h_i\}$ be a maximal family of norm-one elements $h_i \in J$ such that their central supports $c(h_i)$ are mutually orthogonal. Since $A$ is separable, $Z$ is countably decomposable, hence we may enumerate the non-zero central supports as $c(h_n), n \in \mathbb{N}$. Put $h = \sum_{n=1}^{\infty} 2^{-n} h_n \in J$. As $J$ is essential, for a non-zero projection $z \in Z$, there is $h' \in J$ with $h'z \neq 0$. If $hz = 0$ then $c(h)z = 0$ and hence $c(h_n)z = 0$ for all $n \in \mathbb{N}$. It follows that $c(h_n)(c(h')z) \leq c(h_n)z = 0$ which would lead to a contradiction to the maximality assumption on $\{h_n\}$. As a result, $hz \neq 0$ for every non-zero projection $z \in Z$.

3. The second local multiplier algebra

In this section we discuss some necessary and some sufficient conditions for the first and the second local multiplier algebra of a separable $C^*$-algebra $A$ to coincide. The general strategy is that this cannot happen if and only if $M_{loc}(A)$ contains an essential ideal $K$ with the property that $M(K) \setminus M_{loc}(A) \neq 0$.

The following proposition introduces the decisive topological condition in $\text{Prim}(A)$.

**Proposition 3.1.** Let $A$ be a separable $C^*$-algebra such that $\text{Prim}(A)$ contains a dense $G_δ$ subset consisting of closed points. Then $K_A$ is an essential ideal in $M_{loc}(A)$.

**Proof.** Since $K_A$ is a $C^*$-subalgebra of $M_{loc}(A)$, it suffices to show that, whenever $m$ is a multiplier of a closed essential ideal of $A$ and $a \in K_A$, $ma \in K_A$; in fact, we can assume that $a \in A$, by Lemma 2.3.

Let $U \subseteq \text{Prim}(A)$ be a dense open subset and take $m \in M(A(U))$. For $t \in U$, let $\tilde{t} \in \text{Prim}(M(A(U)))$ denote the corresponding primitive ideal under the canonical identification of $\text{Prim}(A)$ with an open dense subset of $\text{Prim}(M(A(U)))$. Let $\{b_n \mid n \in \mathbb{N}\}$ be a countable dense subset of $A$, and let $T$ be the dense $G_δ$ subset $T = \text{Sep}(A) \cap U$. Note that, by Lemma 2.1, there is a dense $G_δ$ subset $T' \subseteq T$ such that $t \mapsto \|(m - b_n)a + \tilde{t}\|$ is continuous for all $n \in \mathbb{N}$ when restricted to $T'$.

Let $\varepsilon > 0$ and take $t \in T'$. Since $A$ is separable and $t$ is a closed point, the canonical mapping $M(A(U)) \to M(A/t)$ is surjective [18, 3.12.10] and, denoting by $\tilde{m}$ the image of $m$ under this mapping, we have $(m - b_n)a + \tilde{t} = (\tilde{m} - (b_n + t))(a + t)$. As $\{b_n + t \mid n \in \mathbb{N}\}$ is dense in $A/t$ and $A/t$ is strictly dense in its multiplier algebra, there is $b_k$ such that $\|(\tilde{m} - (b_k + t))(a + t)\| < \varepsilon$. By the above-mentioned continuity there is therefore an open subset $V \subseteq \text{Prim}(A)$ containing $t$ such that

$$\|(m - b_k)a + \tilde{s}\| < \varepsilon \quad (s \in V \cap T').$$

Letting $z = z_V \in Z$ be the projection from $A(V) + A(V)^⊥$ onto $A(V)$ we conclude that $\|zma - zb_k a\| = \sup_{s \in V \cap T'} \|(m - b_k)a + \tilde{s}\| \leq \varepsilon$. 
We now choose a (necessarily countable) maximal family \( \{ z_k \} \subseteq Z \) of mutually orthogonal projections such that \( \| z_k ma - z_k b_k a \| \leq \varepsilon \) for each \( k \). Then \( \sup z_k = 1 \) and \( \| \sum_k (z_k ma - z_k b_k a) \| \leq \varepsilon \). As \( ma = \sum_k z_k ma \) and \( \sum_k z_k b_k a \in K_A \) we conclude that \( ma \in K_A \) as claimed proving that \( K_A \) is an ideal in \( M_{\text{loc}}(A) \).

In order to show that \( K_A \) is essential let \( y \in M_{\text{loc}}(A) \) be such that \( yK_A = 0 \). Then, in particular, \( yA = 0 \) and thus \( y = 0 \) by [2, Proposition 2.3.3].

The next result was first obtained in [21, Theorem 2.7] but we believe our approach is more direct and more conceptual.

**Theorem 3.2.** Let \( A \) be a separable C*-algebra such that \( \text{Prim}(A) \) contains a dense \( G_δ \) subset consisting of closed points. Then \( M^{(3)}_{\text{loc}}(A) = M^{(2)}_{\text{loc}}(A) \) and coincides with \( M(K_A) \).

**Proof.** Combining Proposition 3.1 with Lemma 2.2 all we need to show is that \( K_I = K_A \) for each \( I \in \mathcal{I}_{\text{loc}}(A) \). Taking \( I \in \mathcal{I}_{\text{loc}}(A) \), the inclusion \( K_I \subseteq K_A \) is evident. Let \( U \subseteq \text{Prim}(A) \) be the open dense subset such that \( I = A(U) \). Let \( T \subseteq \text{Prim}(A) \) be a dense \( G_δ \) subset consisting of closed and separated points. Fix \( a \in A \) and let \( \varepsilon > 0 \). For \( t \in U \cap T \), \((I + t)/t = A(t) \) as \( t \) is a closed point. Therefore there is \( y \in I \) such that \( y + t = a + t \) and hence \( N(a - y)(s) < \varepsilon \) for all \( s \in V \). Letting \( z = z_V \in Z \) be the projection corresponding to \( V \) we obtain \( \| z(a - y) \| \leq \varepsilon \). The same maximality argument as in the proof of Proposition 3.1 provides us with a family \( \{ z_k \} \) of mutually orthogonal projections in \( Z \) and a bounded family \( \{ y_k \} \) in \( I \) with the property that \( \| a - \sum_k y_k z_k \| \leq \varepsilon \). This shows that \( A \subseteq K_I \) and as a result \( K_A \subseteq K_I \) as claimed.

It was shown in [7], see also [4, Section 6], that the C*-algebra \( A = C[0,1] \otimes K(H) \), where \( H = \ell^2 \), has the property that \( M_{\text{loc}}(A) \neq M_{\text{loc}}(M_{\text{loc}}(A)) \). In the following result, we explore a sufficient condition on the primitive ideal space that guarantees this phenomenon to happen.

We shall make use of some topological concepts. Recall that a topological space \( X \) is called **perfect** if it does not contain any isolated points. If the closure of each open subset of \( X \) is open, then \( X \) is said to be *extremally disconnected*. Thus, \( X \) is not extremally disconnected if and only if it contains an open subset which has non-empty boundary. It is a known fact that an extremally disconnected metric space must be discrete.

**Theorem 3.3.** Let \( X \) be a perfect, second countable, locally compact Hausdorff space. Let \( A = C_0(X) \otimes B \) for some non-unital separable simple C*-algebra \( B \). Then \( M_{\text{loc}}(A) \neq M_{\text{loc}}(M_{\text{loc}}(A)) \).

**Proof.** Since every point in \( \text{Prim}(A) = X \) is closed and separated, \( K_A \) is an essential ideal in \( M_{\text{loc}}(A) \), by Proposition 3.1. By Theorem 3.2, \( M_{\text{loc}}(M_{\text{loc}}(A)) = M(K_A) \). To prove the statement of the theorem it suffices to find an element in \( M(K_A) \) not contained in \( M_{\text{loc}}(A) \).

Note that every non-empty open subset \( O \subseteq X \) contains an open subset which has non-empty boundary. This follows from the above-mentioned fact and the assumption that \( O \) is second countable, locally compact Hausdorff and hence metrisable. Therefore, if \( O \) were extremally disconnected, it had to be discrete in contradiction to the hypothesis that \( X \) is perfect.

Let \( \{ V_n \mid n \in \mathbb{N} \} \) be a countable basis for the topology of \( X \). For each \( n \in \mathbb{N} \), choose an open subset \( V_n \) of \( X \) such that \( \overline{V_n} \subseteq V_n \) and \( V_n \) is not open. Put \( W_n = X \setminus \overline{V_n} \). Then \( O_n = V_n \cup W_n \) is a dense open subset of \( X \).
Let $z_n$ denote the equivalence class of $\chi_{V_n} \otimes 1 \in C_b(\mathcal{O}_n, M(B)_\beta) = M(C_0(\mathcal{O}_n) \otimes B)$ in $Z$. Let $(e_n)_{n \in \mathbb{N}}$ be a strictly increasing approximate identity of $B$ with the properties $e_n e_{n+1} = e_n$ and $\|e_{n+1} - e_n\| = 1$ for all $n$; see [16, Lemma 1.2.3], e.g. Put $p_1 = e_1$, $p_n = e_n - e_{n-1}$ for $n \geq 2$. Then $(p_{2n})_{n \in \mathbb{N}}$ is a sequence of mutually orthograd positive norm-one elements in $B$. 

Set $q_n = \sum_{j=1}^{n} z_j p_{2j}$, $n \in \mathbb{N}$, where we identify an element $b \in M(B)$ canonically with the constant function in $M(A) = C_0(X, M(B)_\beta)$. By means of this we obtain an increasing sequence $(q_n)_{n \in \mathbb{N}}$ of positive elements in $M_{\text{loc}}(A)$ bounded by 1. Since the injective envelope is monotone complete [13], the supremum of this sequence exists in $I(A)$ and is a positive element of norm 1, which we will write as $q = \sup_n q_n = \sum_{n=1}^{\infty} z_n p_{2n}$.

Suppose that $q \in M_{\text{loc}}(A)$. Then, for given $0 < \varepsilon < 1/2$, there are a dense open subset $U \subseteq X$ and $m \in C_0(U, M(B)_\beta)$ with $\|m\| \leq 1$ such that $\|m - q\| < \varepsilon$. Upon multiplying both on the left and on the right by $p_{2n}^{1/2}$ we find that

$$\sup_{t \in U \cap \mathcal{O}_n} \|p_{2n}^{1/2} m(t)p_{2n}^{1/2} - \chi_{V_n}(t)p_{2n}^2\| = \|p_{2n}^{1/2} m p_{2n}^{1/2} - z_n p_{2n}^2\| < \varepsilon.$$ 

Let $n \in \mathbb{N}$ be such that $V_n \subseteq U$. Define $f_n \in C_0(U)$ by $f_n(t) = \|p_{2n}^{1/2} m(t)p_{2n}^{1/2}\|$, $t \in U$ (note that $p_{2n}^{1/2} m p_{2n}^{1/2} \in C_0(U, B)$). Then $0 \leq f_n \leq 1$ and

$$f_n(t) - \chi_{V_n}(t) = \|p_{2n}^{1/2} m(t)p_{2n}^{1/2} - \chi_{V_n}(t)p_{2n}^2\| \leq \|p_{2n}^{1/2} m(t)p_{2n}^{1/2} - \chi_{V_n}(t)p_{2n}^2\| < \varepsilon$$

for all $t \in U \cap \mathcal{O}_n$. By construction, $V_n$ is not open; hence $\partial V_n \neq \emptyset$. Each $f_n(t) \in \partial V_n$ also belongs to $W_n \cap V_n'$ as $\partial W_n = \partial V_n$ and hence $f_n(t) \in W_n \cap V_n' \subseteq W_n \cap V_n''$ since $V''_n$ is open. For every $t \in V_n$, $f_n(t) - 1 < \varepsilon$ and hence $f_n(t) \geq 1 - \varepsilon > 1/2$ for all $t \in V_n$, by continuity of $f_n$. In particular, $f_n(t_0) > 1/2$. For every $t \in W_n \cap V_n'$, we have $f_n(t) < \varepsilon < 1/2$ and thus $f_n(t_0) \leq \varepsilon < 1/2$. This contradiction shows that $q \notin M_{\text{loc}}(A)$.

In order to prove that $q$ belongs to $M(K_A)$ it suffices to show that $q a \in K_A$ for every $a \in A$, by Lemma 2.3. For each $n \in \mathbb{N}$ and $a \in A$, $q_n a = a$ since $z_j p_{2j} a \in ZA$. Therefore, $q_n \in M^c(A)$ for each $n$. Note that $^c A$ contains a strictly positive element $h$. Indeed, taking an increasing approximate identity $(q_n)_{n \in \mathbb{N}}$ of $C_0(X)$ we obtain an increasing approximate identity $u_n = q_n \otimes e_n$, $n \in \mathbb{N}$ of $A$. It follows easily that $(u_n)_{n \in \mathbb{N}}$ is an approximate identity for $^c A = \prod \mathbb{N}$. It is well-known that $h = \sum_{n=1}^{\infty} 2^{-n} u_n$ is then a strictly positive element.

As a result, in order to prove that $(q_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $M^c(A)_\beta$, we only need to show that $(q_n h)_{n \in \mathbb{N}}$ is a Cauchy sequence. For $k \in \mathbb{N}$, $p_{2j} \varepsilon_k = (e_{2j} - e_{2j-1}) \varepsilon_k = 0$ if $2j > k + 1$. Consequently,

$$z_j p_{2j} h = \sum_{k=1}^{\infty} 2^{-k} z_j p_{2j} \varepsilon_k = \sum_{k=1}^{\infty} 2^{-k} g_k z_j p_{2j} \varepsilon_k$$

yields that, for each $n \in \mathbb{N}$,

$$q_n h = \sum_{j=1}^{n} \sum_{k=1}^{\infty} 2^{-k} g_k z_j p_{2j} \varepsilon_k$$

$$= \sum_{k=1}^{\infty} 2^{-k} g_k z_1 p_{2} \varepsilon_k + \sum_{k=3}^{\infty} 2^{-k} g_k z_2 p_{4} \varepsilon_k + \ldots + \sum_{k=2n-1}^{\infty} 2^{-k} g_k z_n p_{2n} \varepsilon_k.$$ 

We conclude that, for $m > n$,

$$\|(q_m - q_n) h\| = \sum_{j=n+1}^{m} \sum_{k=2j-1}^{\infty} 2^{-k} g_k z_j p_{2j} \varepsilon_k = \max_{n+1 \leq j \leq m} \sum_{k=2j-1}^{\infty} 2^{-k} g_k z_j \varepsilon_k \leq \sum_{k=2n+1}^{\infty} 2^{-k}$$

since $g_k z_j p_{2j} \varepsilon_k g_k z_k p_{2k} \varepsilon_k = 0$ for all $k, \ell$ whenever $i \neq j$; therefore $\|(q_m - q_n) h\| \to 0$ as $n \to \infty$. This proves that $(q_n)_{n \in \mathbb{N}}$ is a strict Cauchy sequence in $M^c(A)$. Let $\tilde{q} \in M^c(A)$ denotes its limit,
which is a positive element of norm at most one since $M(\beta A)_+$ is closed in the strict topology. In order to show that $\tilde{q} = q$ note at first that $I(M(\beta A)) = I(\beta A) = I(A)$ by [4, Proposition 2.14]. The mutual orthogonality of the $p_{2n}$'s yields $qp_n = qnq_n$ for all $m \geq n$. Thus, for all $a \in \beta A$, $aq_{2n} = aq_nq_n$ which implies that $aq_{2n} = aq_nq_n$ for all $a$. As $A$ is essential in $I(A)$, it follows that $q_n = q_{2n}$ for all $n \in \mathbb{N}$ by [4, Theorem 3.4]. Repeating the same argument using the strict convergence of $(q_n)_{n \in \mathbb{N}}$ we obtain that $q\tilde{q} = \tilde{q}^2$.

For all $1 \leq n \leq m$, $q_n \leq q_m$ and hence $a^*q_n a \leq a^*q_m a$ for every $a \in \beta A$. It follows that, for all $n$, $a^*q_n a \leq a^*\tilde{q}a$ for every $a$ and therefore $q_n \leq \tilde{q}$ for all $n \in \mathbb{N}$. Consequently, $q \leq \tilde{q}$. Together with the above identity $(\tilde{q} - q)\tilde{q} = 0$ this entails that $q = \tilde{q} \in M(\beta A)$.

Finally, for each $a \in A$, we have $qa \in \beta A \subseteq K_A$. This completes the proof. 

**Remark 3.4.** A space $X$ as in Theorem 3.3 is perfect if and only if it contains a dense $G_\delta$ subset with empty interior. In [4, Theorem 6.13], the existence of a dense $G_\delta$ subset with empty interior in the primitive spectrum, which was assumed to be Stonean, was used to obtain a C*-algebra $A$ such that $M_{\text{loc}}(A)$ is a proper subalgebra of $I(A)$ and the latter agreed with $M_{\text{loc}}(M_{\text{loc}}(A))$. In contrast to this example, and also the one considered in [7], our approach in Theorem 3.3 does not need any additional information on the injective envelope; nevertheless all higher local multiplier algebras coincide by Theorem 3.2.

**Remark 3.5.** Taking the two results Corollary 4.8 and Theorem 3.3 together we obtain the following, maybe surprising dichotomy for a compact Hausdorff space $X$ satisfying the assumptions in (3.3). Let $A = C(X) \otimes B$ for a unital, separable, simple C*-algebra $B$. Then $M_{\text{loc}}(A) = M_{\text{loc}}(M_{\text{loc}}(A))$. But if we stabilise $A$ to $A_\delta = A \otimes K(H)$ then $M_{\text{loc}}(A_\delta) \neq M_{\text{loc}}(M_{\text{loc}}(A_\delta))$

With a little more effort we can replace the commutative C*-algebra in Theorem 3.3 by a nuclear one, provided the properties of the primitive ideal space are preserved. We shall formulate this as a necessary condition on a C*-algebra $A$ with tensor product structure to enjoy the property $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$. Note that, whenever $B$ and $C$ are separable C*-algebras and at least one of them is nuclear, the primitive ideal space $\text{Prim}(C \otimes B)$ is homeomorphic to $\text{Prim}(C) \times \text{Prim}(B)$, by [20, Theorem B.45], for example.

Some elementary observations are collected in the next lemma in order not to obscure the proof of the main result.

**Lemma 3.6.** Let $X$ be a topological space, and let $G \subseteq X$ be a dense subset consisting of closed points.

(i) If $X$ is perfect then $G$ is perfect (in itself).
(ii) For each $V \subseteq X$ open, $\overline{V \cap G} = \overline{V \cap G}^G$, where $\overline{G}$ denotes the closure relative to $G$.
(iii) For each $V \subseteq X$ open, $\partial(\overline{V \cap G}^G) = \partial \overline{V \cap G}$.

**Proof.** Assertion (i) is immediate from the density of $G$ and the hypothesis that $X \setminus \{t\}$ is open for each $t \in G$. In (ii), the inclusion "$\supseteq$" is evident. The other inclusion "$\subseteq$" follows from the density of $G$.

To verify (iii), we conclude from (ii) that

$$G \setminus \overline{V \cap G}^G = G \setminus (\overline{V \cap G}) = G \cap (X \setminus \overline{V})$$
and therefore, with $W = X \setminus \overline{V}$,
\[
G \setminus \overline{V} \cap G = G \cap W = G \cap \overline{W},
\]
where we used (ii) another time. This entails
\[
\partial (\overline{V} \cap G) = \overline{V} \cap G \setminus G \setminus \overline{V} = \partial V \cap G
\]
as claimed.

**Theorem 3.7.** Let $B$ and $C$ be separable $C^*$-algebras and suppose that at least one of them is nuclear. Suppose further that $B$ is simple and non-unital and that $\text{Prim}(C)$ contains a dense $G_δ$ subset consisting of closed points. Let $A = C \otimes B$. If $M_{\text{loc}}(A) = M_{\text{loc}}(M_{\text{loc}}(A))$ then $\text{Prim}(C)$ contains an isolated point.

**Proof.** Let $X = \text{Prim}(C) = \text{Prim}(A)$. We shall assume that $X$ is perfect and conclude from this that $M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A)$. By Proposition 3.1, $K_A$ is an essential ideal in $M_{\text{loc}}(A)$. Using Theorem 3.2 we are left with the task to find an element in $M(K_A) \setminus M_{\text{loc}}(A)$.

The hypothesis on $X$ combined with the separability assumption yields a dense $G_δ$ subset $S \subseteq X$ consisting of closed separated points which is a Polish space. By Lemma 3.6 (i), $S$ is a perfect metrisable space and therefore cannot be extremally disconnected, as mentioned before. Since a non-empty open subset of a perfect space is clearly perfect, it follows that every non-empty open subset of $S$ contains an open subset which has non-empty boundary.

Let $\{V_n | n \in \mathbb{N}\}$ be a countable basis for the topology of $X$. For each $n \in \mathbb{N}$, choose an open subset $V_n$ of $X$ such that $V_n = V_n \cap S \subseteq V_n \cap S$ and $V_n \cap S^c$ is not open. By Lemma 3.6 (ii), $V_n \cap S^c = V_n \cap S$ and we shall use the latter, simpler notation. Put $W_n = X \setminus \overline{V_n}$. Then $O_n = V_n \cup W_n$ is a dense open subset of $X$.

Using the same notation as in the fourth paragraph of the proof of Theorem 3.3 we define the element $q \in I(A)$ by $q = \sum_{n=1}^{\infty} z_n \otimes p_{2^n}$. The argument showing that $q \in M(K_A)$ takes over verbatim from the proof of Theorem 3.3. We will now modify the argument in the fifth paragraph of that proof.

Suppose that $q \in M_{\text{loc}}(A)$. For $0 < \varepsilon < 1/4$, there are a dense open subset $U \subseteq X$ and an element $m \in M(A(U))$ with $\|m\| \leq 1$ such that $\|m - q\| < \varepsilon$. Let $n \in \mathbb{N}$ be such that $V_n \subseteq U$ and choose $t_0 \in \partial V_n \cap S \subseteq U \cap S$ using Lemma 3.6 (iii). Since the ideal $C(U)$ of $C$ corresponding to $U$ is not contained in $t_0$, there is $c \in C(U)$ with $\|c\| = 1$ and $\|c - t_0\| = 1$.

As the function $t \mapsto \|c + t\|$ is lower semicontinuous, there is an open subset $V \subseteq U$ containing $t_0$ such that $\|c + t\| > 1 - \varepsilon$ for $t \in V$. Let $\alpha = (c^{1/2} \otimes p_{2^n}^1) \in C(U) \otimes B = A(U)$ and put $f(t) = \|\alpha m + t\|, t \in U$. By [5, Lemma 6.4], $f$ is continuous on $U \cap S$ because $\alpha m \in A$. For each $t \in V \cap O_n$ we have
\[
|f(t) - \chi_{V_n}(t)| \leq \|\alpha m + t - \chi_{V_n}(t)\| c + t\| + \|\chi_{V_n}(t)\| \|c + t\| - \chi_{V_n}(t)|
\leq \|\alpha m + t - \chi_{V_n}(t)c \otimes p_{2^n}^1 + t\| + (1 - \|c + t\|) \chi_{V_n}(t)
\leq \|\alpha m - q\| + \varepsilon < 2 \varepsilon,
\]
since $(c^{1/2} \otimes p_{2^n}^1) q (c^{1/2} \otimes p_{2^n}^1) = cz_n \otimes p_{2^n}^2$. For each $t \in V_n \cap S$ we have $f(t) > 1 - 2 \varepsilon > 1/2$ and therefore $f(t_0) > 1/2$ by continuity of $f$ on $U \cap S$ and the fact that $V_n \cap S^c = V_n \cap S$ by Lemma 3.6 (ii), thus $t_0 \in \partial V_n \cap S^c$.

On the other hand,
\[
t_0 \in \overline{W_n} \cap V \cap S \subseteq \overline{W_n} \cap V \cap S = \overline{W_n} \cap V \cap S^c.
\]
as $V$ is open and using Lemma 3.6 (ii) again. Thus $f_n(t_0) \leq 2 \varepsilon < 1/2$. This contradiction shows that $q \notin M_{loc}(A)$, and the proof is complete. \hfill \Box

We can now formulate an if-and-only-if condition characterising when the second local multiplier algebra is equal to the first.

**Corollary 3.8.** Let $A = C \otimes B$ for two separable $C^*$-algebras $B$ and $C$ satisfying the conditions of Theorem 3.7. Suppose that $\text{Prim}(A)$ contains a dense $G_δ$ subset consisting of closed points. Then $M_{loc}(A) = M_{loc}(M_{loc}(A))$ if and only if $\text{Prim}(A)$ contains a dense subset of isolated points.

**Proof.** Let $X = \text{Prim}(A)$, $X_1$ the set of isolated points in $X$ and $X_2 = X \setminus \overline{X}_1$. Then $X_1$ and $X_2$ are open subsets of $X$ with corresponding closed ideals $I_1 = A(X_1)$ and $I_2 = A(X_2)$ of $A$. If $X_1$ is dense, $I_1$ is the minimal essential closed ideal of $A$ so $M_{loc}(A) = M(I_1)$. It follows that

$$M_{loc}(M_{loc}(A)) = M_{loc}(M(I_1)) = M_{loc}(I_1) = M_{loc}(A).$$

In the general case, $M_{loc}(A) = M_{loc}(I_1) \oplus M_{loc}(I_2)$ by [2, Lemmas 3.3.4 and 3.3.6]. If $X_2 \neq \emptyset$, it contains a dense $G_δ$ subset of closed points and so $I_2 = C(X_2) \otimes B$ satisfies all the assumptions in Theorem 3.7 while $X_2$ is a perfect space. It follows that

$$M_{loc}(M_{loc}(A)) = M_{loc}(M_{loc}(I_1) \oplus M_{loc}(I_2)) = M_{loc}(M_{loc}(I_1)) \oplus M_{loc}(M_{loc}(I_2))$$

$$\neq M_{loc}(I_1) \oplus M_{loc}(I_2) = M_{loc}(A). \quad \Box$$

4. A sheaf-theoretic approach

In [5], we developed the basics of a sheaf theory for general $C^*$-algebras. This enabled us to establish the following formula for $M_{loc}(A)$ in [5, Theorem 7.6]:

$$M_{loc}(A) = \text{alg lim}_{T \in T} \Gamma_b(T, A),$$

where $A$ is the upper semicontinuous $C^*$-bundle canonically associated to the multiplier sheaf $M_A$ of $A$ [5, Theorem 5.6] and $\Gamma_b(T, A)$ is the $C^*$-algebra of bounded continuous sections of $A$ on $T$. A like description is available for the injective envelope:

$$I(A) = \text{alg lim}_{T \in T} \Gamma_b(T, I),$$

where the $C^*$-bundle $I$ corresponds to the injective envelope sheaf $\mathcal{J}_A$ of $A$, see [5, Theorem 7.7]. These descriptions are compatible, by [5, Corollary 7.8]. Since a continuous section is determined by its restriction to a dense subset, the $*$-homomorphisms $\Gamma_b(T, B) \rightarrow \Gamma_b(T', B)$, $T' \subseteq T$, $T' \in T$ are injective for any $C^*$-bundle $B$ and thus isometric. Consequently, an element $y \in M_{loc}(M_{loc}(A))$ is contained in some $C^*$-subalgebra $\Gamma_b(T, I)$ and will belong to $M_{loc}(A)$ once we find $T' \subseteq T$, $T' \in T$ such that $y \in \Gamma_b(T', A)$.

**Remark 4.1.** Let $a \in \Gamma_b(T, A)$ for a separable $C^*$-algebra $A$. By applying Lemma 2.1 to the negative of the upper semicontinuous norm function on $A$, there is always a smaller dense $G_δ$ subset $S \subseteq \text{Sep}(A) \cap T$ on which the restriction of the function $t \mapsto \|a(t)\|$ is continuous.

On the basis of this, we shall obtain a concise proof of an extension of one of Somerset’s main results, [21, Theorem 2.7], in this section. This extension is twofold: firstly, we replace the
assumption of an identity by the more general hypothesis on $A$ to be quasicentral. Secondly, we establish the result for $C^*$-subalgebras of $M_{loc}(A)$ containing $A$.

The following concept was introduced and initially studied by Delaroche [9], [10]. A $C^*$-algebra $A$ is called \emph{quasicentral} if no primitive ideal of $A$ contains the centre $Z(A)$ of $A$. We recall some basic properties of quasicentral $C^*$-algebras.

\begin{remark}
Let $A$ be a quasicentral $C^*$-algebra.
\begin{enumerate}[(i)]
\item The mapping $\nu : \text{Prim}(A) \to \text{Max}(Z(A))$, $t \mapsto t \cap Z(A)$ is well-defined, surjective and continuous.
\item The Dauns–Hofmann isomorphism $Z(M(A)) \to C_0(\text{Prim}(A))$, $z \mapsto f_z$ such that $za + t = f_z(t)(a + t)$ for all $a \in A$, $z \in Z(M(A))$ and $t \in \text{Prim}(A)$ maps $Z(A)$ onto $C_0(\text{Prim}(A))$; see [20, A.34] and [9, Proposition 1].
\item Every approximate identity of $Z(A)$ is an approximate identity for $A$ and thus $A = Z(A)A$; see [6, Proposition 1].
\end{enumerate}
\end{remark}

Part (i) of the result below on the existence of local identities is already contained in [9, Proposition 2] but we provide an independent brief proof along the lines of the proof of [6, Theorem 5].

\begin{lemma}
Let $A$ be a quasicentral $C^*$-algebra, $C \subseteq \text{Prim}(A)$ compact and $t \in C$.
\begin{enumerate}[(i)]
\item There exists $z \in Z(A)_+$, $\|z\| = 1$ such that $z + s = 1_{A/s}$, the identity in the primitive quotient $A/s$ for all $s \in C$.
\item Let $U_1$ be an open neighbourhood of $t$ contained in $C$ and let $U_2 = \text{Prim}(A) \setminus \overline{U_1}$. If $z \in Z(A)_+$ is as in (i) then $z + A(U_2)$ is the identity in $A/A(U_2)$.
\end{enumerate}
\end{lemma}

\begin{proof}
As $\text{Max}(Z(A))$ is a locally compact Hausdorff space, there is $f \in C_0(\text{Max}(Z(A)))_+$ with $\|f\| = 1$ such that $f(\nu(s)) = 1$ for all $s \in C$ [19, 1.7.5]. Identifying $Z(A)$ with $C_0(\text{Prim}(A))$, see Remark 4.2 above, we obtain $z \in Z(A)_+$, $\|z\| = 1$ such that $f_z = f \circ \nu$ and hence $z + s = 1_{A/s}$ for all $s \in C$. This proves (i).

Now let $U_1$ be an open neighbourhood of $t$ contained in $C$ and put $U_2 = \text{Prim}(A) \setminus \overline{U_1}$. Let $z \in Z(A)_+$ be as in (i). Then $\overline{U_1} = \{s \in \text{Prim}(A) \mid A(U_2) \subseteq s\}$ is homeomorphic to $\text{Prim}(A/A(U_2))$ via $s \mapsto s/A(U_2)$ [18, 4.1.11]. Therefore, any identity which holds in $(A/A(U_2))/s/A(U_2))$ for a dense set of $s$ holds in $A/A(U_2)$. Since $(A/A(U_2))/s/A(U_2)) \cong A/s$ and $z + s = 1_{A/s}$ for all $s \in U_1$, it follows that $z + A(U_2) = 1_{A/A(U_2)}$ as claimed in (ii).
\end{proof}

With the help of Lemma 4.3 we can extend a key result in [5], viz. [5, Lemma 6.9], from the unital case to the situation of quasicentral $C^*$-algebras. Though the proof is similar, we include the details for completeness.

\begin{proposition}
Let $A$ be a quasicentral $C^*$-algebra, and let $t \in \text{Prim}(A)$ be a closed and separated point. Then the natural mapping $\varphi_t : A_t \to A/t$ is an isomorphism.
\end{proposition}

\begin{proof}
Since $A$ is quasicentral, the $C^*$-algebra $A/t$ is unital, and since $t$ is a closed point, $A/t$ is simple. Therefore the natural mapping $\varphi_t : A_t \to M_{loc}(A/t)$ given by [5, Proposition 6.2] simplifies to $\varphi_t : A_t \to A/t$. As $t$ is a separated point, $\ker t_t = t$ where $t_t : A \to A_t$ is the canonical map [5, Proposition 6.5]. Since $\varphi_t \circ t_t = \pi_t$, where $\pi_t$ is the canonical surjection $A \to A/t$, we find that $\varphi_t$ is injective when restricted to $t_t(A)$.
\end{proof}
Let $U$ be an open neighbourhood of $t$ in $\text{Prim}(A)$, and take $m \in M(A(U))$. Since $N(z - e)(t) = 0$ and $N(z - e)$ is continuous at $t$, as $t$ is a separated point [5, Lemma 6.4], there is an open neighbourhood $U_1$ of $t$ contained in $C$ such that $N(z - e)(s) < 1/2$ for every $s \in U_1$. Set $Y = U_1$ and $U_2 = \text{Prim}(A) \setminus Y$. By Lemma 4.3 (ii), $z + A(U_2)$ is the identity of $A/A(U_2)$. Since $A(U_1)$ sits as an essential ideal in $A/A(U_2)$, we have an embedding of unital $C^*$-algebras $A/A(U_2) \subseteq M(A(U_1)) = \mathfrak{M}_A(U_1)$. The set $\{s \in \text{Prim}(A) | N(z - e)(s) \leq 1/2\}$ is closed in $\text{Prim}(A)$ and contains $U_1$; consequently $N(z - e)(s) \leq 1/2$ for every $s \in Y$.

Since $N_{A/A(U_2)}((z - e) + A(U_2))(s) = N_A(z - e)(s) \leq 1/2$ for every $s \in Y$, we get that $\|1_{A/A(U_2)} - e + A(U_2)\| = \|(z - e) + A(U_2)\| \leq 1/2 < 1$, and thus $e + A(U_2)$ is invertible in $A/A(U_2)$. Take any $y \in A$ such that $y + A(U_2) = (e + A(U_2))^{-1}$. Then we have

$$m_{\mathfrak{M}_A(U_1)} = m_{\mathfrak{M}_A(U_1)}(e + A(U_2))(y + A(U_2)) = (me + A(U_2))(y + A(U_2)) \in A/A(U_2),$$

since $me \in A(U) \subseteq A$. As a result, $m_{\mathfrak{M}_A(U_1)}$ belongs to the image of the map $A \to \mathfrak{M}_A(U_1)$. We thus find that the image of $m$ in $\mathcal{A}_t = \varinjlim \mathfrak{M}_A(W)$ belongs to the image of the map $A \to \mathcal{A}_t$, and it turns out that the map $A/t \to \mathcal{A}_t$ is surjective. Since it is also injective, we conclude that it is an isomorphism, and so its inverse, $\varphi_t$, must be an isomorphism too.

The following example shows that the statement of Proposition 4.4 can fail if the $C^*$-algebra is not quasicentral.

**Example 4.5.** Let $B = C_0(\mathbb{N}, M_2(\mathbb{C}))$ be the $C^*$-algebra of all bounded (continuous) functions from $\mathbb{N}$ to the $2 \times 2$ complex matrices. We shall write elements of $B$ as $x = (x(n))_{n \in \mathbb{N}}$. Let $A$ be the $C^*$-subalgebra of $B$ consisting of those $x$ such that $x_{ij}(n) \to 0$, $n \to \infty$ for $(i, j) \neq (1, 1)$ and $x_{11}(n) \to \mu(x)$, $n \to \infty$. Then $A$ is a non-unital separable $2$-subhomogeneous $C^*$-algebra with Hausdorff primitive spectrum. In fact, the primitive ideals of $A$ are given by $t_{\infty} = \ker \mu$ and, for each $n \in \mathbb{N}$, $t_n = \{x \in A | x(n) = 0\}$ (with corresponding irreducible representations given by $\pi_{\infty}: A \to \mathbb{C}$, $\pi_{\infty}(x) = \mu(x)$ and $\pi_n: A \to M_2(\mathbb{C})$, $\pi_n(x) = x(n)$, $x \in A$). Clearly $\text{Prim}(A)$ is homeomorphic to the one-point compactification $\mathbb{N}_{\infty}$ of $\mathbb{N}$, since $\{U_n | n \in \mathbb{N}\}$ with $A(U_n) = \bigcap_{j=1}^{\infty} t_j$ forms a neighbourhood basis for $t_{\infty}$.

As $C_0(\mathbb{N}, M_2(\mathbb{C})) = t_{\infty}$, $t_{\infty}$ is an essential ideal of $A$ and $M_{\text{loc}}(t_{\infty}) = M(t_{\infty}) = B = I(A)$. Moreover, $M(A)$ consists of those $x$ satisfying $\lim_n x_{12}(n) = \lim_n x_{21}(n) = 0$, $\lim_n x_{11}(n) = \mu(x)$, and $(x_{22}(n))_{n \in \mathbb{N}}$ is bounded. It follows that $\text{Prim}(M(A)) = \beta\mathbb{N} \cup \{t_{\infty}\}$, where all the ultrafilters in $\beta\mathbb{N}$ yield characters of $M(A)$ via $\lim_n x_{22}(n)$. Any open neighbourhood of $t_{\infty}$ in $\text{Prim}(M(A))$ must contain one of the $U_n$’s and hence $t_m$ for $m \geq n + 1$. As $\mathbb{N}$ is dense in $\beta\mathbb{N}$ we conclude that no point in $\beta\mathbb{N} \setminus \mathbb{N}$ can be separated from $t_{\infty}$.

This leads to the following description of the associated upper semicontinuous $C^*$-bundle. For each $n \in \mathbb{N}$, $\mathcal{A}_n \cong A/t_n = M_2(\mathbb{C})$. On the other hand, $\mathcal{A}_\infty = \varinjlim_i M(\mathcal{A}(U_n))$ with the connecting mappings given by

$$(0, \ldots, 0, y(n + 1), y(n + 2), \ldots) \mapsto (0, \ldots, 0, 0, y(n + 2), \ldots)$$

taking into account that $\mathcal{A}_\infty \cong M(A)$ for each $n$. It follows that $\mathcal{A}_\infty$ is indeed commutative and isomorphic to $C(\{t_{\infty}\} \cup \beta\mathbb{N} \setminus \mathbb{N}) = C \times \ell^\infty/c_0$. As a result, the homomorphism $\varphi_{t_{\infty}}: \mathcal{A}_\infty \to A/t_{\infty} = C$ is far from being injective. Note that $t_{\infty} \supseteq Z(A) \cong c_0$ so that $A$ is not quasicentral.

To complete the picture we note that, in the $C^*$-bundle $I$ associated to the injective envelope sheaf, the fibres are $I_n = M_2(\mathbb{C})$, $n \in \mathbb{N}$ and $I_\infty = M_2(\ell^\infty/c_0)$ with the embedding $\mathcal{A}_\infty \to I_\infty$ simply the diagonal map.
A quasicentral C*-algebra $A$ is said to be central if the mapping $\nu$ of Remark 4.2 (i) is injective. Since this is equivalent to the hypothesis that $A$ has Hausdorff primitive spectrum [9, Proposition 3], the same arguments as in Theorem 6.10 and Corollary 6.11 of [5] yield the following consequence.

**Corollary 4.6.** Let $A$ be a central separable C*-algebra. Then all the fibres $A_t = A/t$, $t \in \text{Prim}(A)$ are isomorphic to the fibres $A_t$ associated to the multiplier sheaf $\mathcal{M}_A$ of $A$. Indeed, the multiplier sheaf $\mathcal{M}_A$ of $A$ is isomorphic to the sheaf $\Gamma_b(\cdot, A)$ of bounded continuous local sections of the C*-bundle $A$ associated to $\mathcal{M}_A$.

Every C*-algebra $A$ contains a largest quasicentral ideal $J_A$, which is the intersection of all closed ideals in $A$ that contain $Z(A)$ [10, Proposition 1]. Clearly, the hypothesis in our main result of this section below is equivalent to the assumption that $J_A$ is essential.

**Theorem 4.7.** Let $A$ be a separable C*-algebra such that $\text{Prim}(A)$ contains a dense $G_δ$ subset consisting of closed points. Suppose $A$ contains a quasicentral essential closed ideal. If $B$ is a C*-subalgebra of $M_{\text{loc}}(A)$ containing $A$ then $M_{\text{loc}}(B) \subseteq M_{\text{loc}}(A)$. In particular, $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$.

**Proof.** As $M_{\text{loc}}(I) = M_{\text{loc}}(A)$ for every $I \in \mathcal{I}_{\text{ce}}(A)$, we can assume without loss of generality that $A$ itself is quasicentral.

Take $y \in M(J)$ for some $J \in \mathcal{I}_{\text{ce}}(B)$, and let $T \in \mathcal{T}$ be such that $y \in \Gamma_b(T, 1)$ (recall that $M_{\text{loc}}(B) \subseteq I(B) = I(A)$). By hypothesis, and the fact that Sep($A$) itself is a dense $G_δ$ subset, we can assume that $T$ consists of closed separated points of $\text{Prim}(A)$. Take $h \in J$ with the property that $hz \neq 0$ for every non-zero projection $z \in Z$ (Lemma 2.4). By Remark 4.1, there is $S \in \mathcal{T}$ contained in $T$ such that the function $t \mapsto \|h(t)\|$ is continuous when restricted to $S$ (viewing $h$ as a section in $\Gamma_b(S, A)$). Consequently, the set $S' = \{t \in S \mid h(t) \neq 0\}$ is open in $S$ and intersects every $U \subseteq \mathcal{T}$ non-trivially; it is thus a dense $G_δ$ subset of $\text{Prim}(A)$. Replacing $T$ by $S'$ if necessary, we may assume that $h(t) \neq 0$ for all $t \in T$.

A standard argument yields a separable C*-subalgebra $B'$ of $J$ containing $AhA$ and such that $yB' \subseteq B'$ and $B'y \subseteq B'$. Let $\{b_n \mid n \in \mathbb{N}\}$ be a countable dense subset of $B'$. For each $n$, let $T_n \in \mathcal{T}$ be such that $b_n \in \Gamma_b(T_n, A)$. Letting $T' = \bigcap_{n} T_n \cap T \in \mathcal{T}$ we find that $B' \subseteq \Gamma_b(T', A)$ and hence $B'_t = \{b(t) \mid b \in B'\} \subseteq A_t$ for each $t \in T'$.

For each $t \in T$, the C*-algebras $A_t$ and $A/t$ are isomorphic, by Proposition 4.4 above, and since $A/t$ is unital and simple (as $t$ is closed), we obtain $A_t, h(t)A_t = A_t$ for each $t \in T'$. Consequently,

$$A_t = A_t h(t)A_t = (A/t)h(t)(A/t) = A_t h(t)A_t = (AhA)_t \subseteq B'_t,$$

and thus $B'_t = A_t$ for all $t \in T'$. We can therefore find, for each $t \in T'$, an element $b_t \in B'$ such that $b_t(t) = 1(t)$. It follows that $y(t) = y(t) 1(t) = (yb_t)(t) \in A_t$ for all $t \in T'$, which yields $y \in \Gamma_b(T', A)$. This proves that $y \in M_{\text{loc}}(A)$. \hfill $\square$

**Corollary 4.8.** For every central separable C*-algebra $A$, $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$.

In [17], Pedersen showed that every derivation of a separable C*-algebra $A$ becomes inner in $M_{\text{loc}}(A)$ when extended to the local multiplier algebra. His question whether every derivation of $M_{\text{loc}}(A)$ is inner (when $A$ is separable) has since been open and seems to be connected to the problem how much bigger $M_{\text{loc}}(M_{\text{loc}}(A))$ can be. In this direction, Somerset proved the
COROLLARY 4.9. Let $A$ be a quasicentral separable C*-algebra such that \( \text{Prim}(A) \) contains a dense $G_δ$ subset consisting of closed points. Then every derivation of $M_{\text{loc}}(A)$ is inner.

\[ \text{Proof.} \] Let \( d : M_{\text{loc}}(A) \to M_{\text{loc}}(A) \) be a derivation. Let $B$ be a separable C*-subalgebra of $M_{\text{loc}}(A)$ containing $A$ which is invariant under $d$. By [2, Theorem 4.1.11], \( d_B = d_{\text{loc}}B \) can be uniquely extended to a derivation $d_{M_{\text{loc}}(B)} : M_{\text{loc}}(B) \to M_{\text{loc}}(B)$. Both derivations can be uniquely extended to their respective injective envelopes, by [14, Theorem 2.1], but since $I(B) = I(M_{\text{loc}}(B))$, we have $d_I(B) = d_I(M_{\text{loc}}(B))$. The same argument applies to the extension of $d$, since $I(B) = I(A) = I(M_{\text{loc}}(A))$; in other words, $d_I(M_{\text{loc}}(A)) = d_I(B)$ which we will abbreviate to $d$. By [17, Proposition 2], $d_{M_{\text{loc}}(B)} = ad_y$ for some $y \in M_{\text{loc}}(B)$; in fact, $y \in M_{\text{loc}}(A)$ by Theorem 4.7. By uniqueness, $d = ad_y$ and hence $d = ad_y$ on $M_{\text{loc}}(A)$. \[ \square \]

References