In network models for dynamic traffic assignment the link travel times are often described by “whole-link” models. In particular, they have been expressed as a function of the number of vehicles currently on the link. Such whole-link models are useful approximations and make the network model tractable when flows and travel times are varying over time. Here we propose an alternative whole-link model to approximate travel times. For a vehicle entering a link at time $t$, we let the link travel time be a function of a weighted average of the inflow rate at the time it enters and the outflow rate at the time it exits. We show that this model ensures first-in-first-out and has other desirable properties. We indicate computational methods for solving the model, apply it to various patterns of inflows, and compare the numerical results with two alternative whole-link models.

If the inflow to a link is varying over time, then to accurately model the link travel times or outflows varying over time would usually require modeling the flow, speed, and density evolving along the link. However, even when link inflows are varying over time, it has often been found convenient to express, or approximate, the outflows or travel times as a function of variables relating to the whole link, for example, as a function of the number of vehicles on the link. This is common in network models for dynamic traffic assignment, particularly those formulated as mathematical programs. Following Heydecker and Addison (1998) we refer to such link models as “whole-link” models, to distinguish them from modeling the flow, speed, and density evolving along the link. To ensure a reasonable or acceptable approximation to real traffic behavior, whole-link models should satisfy at least certain properties stated below. Finding models that have such properties has proven elusive, and it is known that some whole-link models in use or that are proposed in the literature may not satisfy these desirable properties. In this paper we are concerned with developing an alternative whole-link model that satisfies these properties.

We are not concerned in this paper with the overall arguments for or against using whole-link travel-time models in dynamic traffic assignment for networks, nor with issues that may arise in embedding such link travel-time models in a dynamic network model. These issues also arise with other link travel-time models and are not the focus of this paper. Here we simply note that such whole-link models are widely used in the DTA network models in the current literature.

The following three properties (stated in no particular order) are widely considered desirable for travel-time functions with time-varying flows.

**Property 1 (First-in-first-out (FIFO)).** This can be stated in various equivalent ways, for example, traffic that enters a link up to any time $t$ will exit from the link before traffic that enters after time $t$. This is not intended to preclude individual vehicles, travelling in the same direction, overtaking and passing each other. Hence, for traffic entering at say time $t$, the exit time can be interpreted as an...
"average" exit time, or the exit time for an "average" vehicle. Without a FIFO condition, all traffic entering at time $t$ could exit before traffic that entered much earlier than time $t$, even though this is usually not physically possible for traffic on a single link.

Property 2 (Causality). The link travel times (and hence outflow rates) for traffic entering at time $t$ depend on the traffic entering at time $t$ and earlier than $t$, but not on traffic entering later than $t$. This is sometimes referred to as a causality property. Note that this is not necessary or sufficient for FIFO. For example, suppose the travel time for traffic entering at time $t$ depends only on the inflow rate at time $t$. This satisfies causality but can violate FIFO.

Property 3 (Reduction to a static model). When traffic flows are constant over time the link travel-time function should reduce to the well-known static model. In the latter, traffic flow, speed, density, and travel time are all assumed constant, and the model can be expressed in various equivalent forms, flow-density, speed-density, travel-time-flow, and so on.

There are also other desirable properties that are more qualitative or more difficult to define exactly. For example, it is desirable that satisfying Properties 1 to 3 above should not impose unrealistic or unacceptable restrictions on parameter values. It is desirable that the parameters of the model should be amenable to being estimated from available data.

Existing Whole-Link Travel-Time Models
Friesz et al. (1993), §3 introduce the following linear travel-time model for use in a network model for dynamic traffic assignment:

$$
\tau(t) = a + bx(t)
$$

(1)

where $a$ and $b$ are constants, $x(t) = \int_0^t (u(s) - v(s)) \, ds$ is the number of vehicles on the link at time $t$, and $u(t)$ and $v(t)$ are the inflow and outflow, respectively, for the link at time $t$. They show that this satisfies FIFO for all continuous inflow patterns $u(t)$. Others (e.g., Ran et al. 1993, Ran and Boyce 1996) have sometimes used more general nonlinear whole-link travel-time functions of the form

$$
\tau(t) = g(x(t), u(t), v(t)).
$$

(2)

They have used special cases of this, such as $\tau(t) = g_1(x(t), u(t))$ or $\tau(t) = g_2(x(t), v(t))$ where $g_1(\cdot)$ and $g_2(\cdot)$ respectively relate to the travel part of the link and the queue at the exit from the link. Daganzo (1995) draws attention to problems with the general form (2). He shows that a sufficiently fast decline in the inflows $u(t)$ will make model (2) violate FIFO, hence he recommends omitting $u(t)$. He also shows that a jump in inflow can cause a later jump in outflow and hence an unrealistic jump in travel time; he hence recommends omitting $v(t)$ from model (2), which leaves

$$
\tau(t) = g(x(t)).
$$

(3)

He does not discuss whether the latter satisfies FIFO.

The model (1), and the nonlinear version (3), has been further considered and investigated by several authors, including Astarita (1995, 1996), Wu et al. (1995, 1998), Xuet al. (1999), Carey and McCartney (2002), and Zhu and Marcotte (2000), and has been used in other network models, such as Adamo et al. (1999). For example, Xuet al. (1999) consider both the linear and nonlinear form of $\tau(t) = g(x(t))$ and show that these satisfy FIFO under certain weak conditions, in particular inflows bounded from above and, in the nonlinear case, the gradient of $g(x)$ bounded from above. The function $\tau(t) = g(x(t))$ has sometimes been used to represent the travel time along a “travel link” and sometimes used to represent the waiting time in a queue or “queuing link.” Use of the model $\tau(t) = g(x(t))$ has not been without criticism. If it is used to represent the travel time on a link, then $\tau(t) = g(x(t))$ implies that the link travel time is independent of distribution of traffic along the link. As a result, if the inflow rate is changing rapidly, over a time span that is less than the time taken to traverse the link, then the resulting travel times may be unrealistic. However, it should be noted that whole-link models are not intended to accurately describe such rapid changes in traffic flows.

1. A Continuous Whole-Link Travel-Time Model to Satisfy FIFO

In the travel-time function (2), all variables relate to the same point in time, namely $t$. In particular, for
traffic entering the link at time $t$, the travel time depends on the outflow rate at that time. A more natural assumption is that, for traffic entering at time $t$, the travel time is related to the inflow rate when it enters (at time $t$) and the outflow rate at the time it exits from the link, at time $t + \tau(t)$. Thus, let

$$\tau(t) = h(u(t), v(t + \tau(t))), \quad (4)$$

instead of $\tau(t) = h(u(t), v(t))$. This seems more natural because it is the flow rate $v(t + \tau(t))$ and not $v(t)$ that the vehicles actually experience when exiting. We will show that, for properly chosen forms of $h(\cdot, \cdot)$, (4) satisfies the three properties set out above, in particular FIFO.

More specifically, we introduce a new whole-link travel-time model as follows. If the flow rate is constant along a link, then the link travel time is given by the well-known static travel-time function $\tau = f(u)$, which is easily obtained from the flow-density equation or the speed-flow equation. In this paper, as in static assignment models, we consider only a nondecreasing travel-time function $f(u)$, without a backward bending part. This corresponds to a nondecreasing flow-density function and a nonincreasing speed-flow function. Thus we consider only flows that are described as uncongested in the traffic engineering literature and are described as congested, but not hypercongested, in the economics literature (e.g., see Lindsey and Verhoef 2000). Now consider flow varying along the link. In that case, as an approximation, we propose using this same travel-time function $f(\cdot)$ but applying it to the average flow rate $w(t)$ experienced by a vehicle traversing the link, having entered the link at time $t$. Here, as an estimate of $w(t)$ we use a weighted average of the flow rate $u(t)$ that the vehicle experiences when entering the link at time $t$ and the flow rate $v(t + \tau(t))$ that it experiences when exiting from the link at time $t + \tau(t)$. Thus,

$$w(t) = \beta u(t) + (1 - \beta)v(t + \tau(t)), \quad (5)$$

where $\beta$ is a weighting constant $1 \geq \beta \geq 0$. If the flow rate increases (or decreases) linearly along a link, from the entrance to the exit, then the average flow rate (at the midpoint of the link) is obtained by letting $\beta = 1/2$ in (5). In numerical examples we used $\beta = 1/2$. We can now write the link travel time as

$$\tau(t) = f(w(t)) = f(\beta u(t) + (1 - \beta)v(t + \tau(t))). \quad (6)$$

As usual, we assume that $f(\cdot)$ is nondecreasing. Note that if we let $\beta = 1$, then (6) reduces to $\tau(t) = f(u(t))$, so that the travel time is a function only of inflows, as is used in static traffic assignment models. If we let $\beta = 0$, then (6) reduces to $\tau(t) = f(v(t + \tau(t)))$, so that the travel time is a function only of the outflow at the time of exit. To derive desirable properties for the above model we will usually assume or require $1 > \beta > 0$, though in some results (for the sake of generality) we allow $1 \geq \beta$ or $\beta \geq 0$.

To compute travel times from (6) we need the outflow rate $v(t + \tau(t))$. For this we use a well-known equation:

$$v(t + \tau(t)) = \frac{u(t)}{1 + \tau'(t)}, \quad (7)$$

where $\tau'(t)$ denotes $d\tau(t)/dt$. This equation has been shown by various authors (e.g., Astarita 1995) to hold when FIFO holds and flows are conserved along the link. Hence Equation (7) is necessary for FIFO (assuming flow conservation), but it is not sufficient to ensure FIFO. Some authors may have implied that (7) is sufficient for FIFO, but to show it is sufficient they have assumed that $v(t + \tau(t)) \geq 0$. However, if (7) holds, then assuming $v(t + \tau(t)) > 0$ is equivalent to assuming $\tau'(t) < -1$, hence assuming FIFO. (To see this, note that if $v(t + \tau(t)) > 0$ holds and $u(t) > 0$, then the r.h.s. and l.h.s. of (7) are positive, hence $\tau'(t) > -1$. Conversely, if $\tau'(t) > -1$, then (7) implies $v(t + \tau(t)) > 0$.) It might seem reasonable to assume $v(t + \tau(t)) \geq 0$, or assume traffic flows in the “right direction,” because this occurs in practice for real traffic. However, here we are not dealing with real traffic, but with a mathematical model, and in a mathematical model if the traffic entering the link at time $t$ violates FIFO, it can yield a negative outflow rate at time $t + \tau(t)$, even if this does not have a useful interpretation in practice. The above note, that (7) is not sufficient to ensure FIFO, is important. It means that by including (7) in our model below, we do not ensure FIFO: We still have to prove that the model satisfies FIFO and yields $v(t + \tau(t)) \geq 0$. 

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**Travel-Time Model**
The proposed whole-link travel-time model now consists of (6) and (7). To solve (6)–(7), and to derive its properties, it is convenient to use (7) to substitute for \( v(t + \tau(t)) \) in (6) to give a link travel-time model

\[
\tau(t) = f(w(t)) = f \left( \beta u(t) + (1 - \beta) \frac{u(t)}{1 + \tau(t)} \right) \tag{8}
\]

then invert and rearrange to give an equivalent form

\[
\tau'(t) = -1 + \frac{(1 - \beta)u(t)}{f^{-1}(\tau) - \beta u(t)}. \tag{9}
\]

Equation (9) is of the form \( \tau'(t) = \phi(\tau, t) \) and is a first-order ordinary differential equation. Even when the forms of \( f(\cdot) \) and \( u(\cdot) \) are given, (9) usually can not be solved for \( \tau \) as an explicit function of \( t \), but can be solved numerically (see §2). A solution of (9) is a \( \tau = \varphi(t) \), \( t_0 \leq t \leq T \), that satisfies (9) and a given initial condition \( \tau_0 = \varphi(t_0) \).

**Proposition 1.** The model (6)–(7) satisfies Properties 2 and 3 from the introduction. Similarly for the model (8), or equivalently (9).

**Proof.** First consider Property 2: The time argument of all variables in (6) and (7) is \( t \), except for \( v(t + \tau(t)) \). However \( v(t + \tau(t)) \) is defined in (7), in terms of variables with time argument \( t \). Hence, for traffic entering at time \( t \), the travel time \( \tau(t) \) depends on inflows at time \( t \). Further, (6)–(7) implies (9). A solution for the differential Equation (9) depends on the initial conditions (at time \( 0 \)) and on the parameter \( u(t) \) from time \( 0 \) to \( t \). Hence, \( \tau(t) \) depends on \( u(t) \) from Time \( 0 \) to \( t \). However \( \tau(t) \) is the integral of \( \tau'(t) \), hence it also depends on \( u(t) \) from time \( 0 \) to \( t \).

Now consider Property 3: If flows are constant over time, then \( u(t) = v(t) = v(t + \tau(t)) \), hence (6) reduces to \( \tau = f(u) \). If flow and travel time are constant over time, then \( \tau(t) = 0 \) and (8) reduces to \( \tau = f(u) \). □

As noted in the paragraph containing (7), we still have to prove that the model (6)–(7) satisfies FIFO (Property 1 in the introduction). We can show this, under weak conditions on \( f(\cdot) \) and \( u(\cdot) \), for the model stated as (6), (7)) or as (8) or (9). We first derive FIFO and other properties for (9). Because (9) was obtained from (6)–(7), all solutions of (9) satisfy (6)–(7), which does not mean that all solutions of (6)–(7), or its properties, can be recovered from (9). However, the following proposition ensures that the relevant properties of (9) apply also to (6)–(7).

**Proposition 2.** If (9) holds and satisfies FIFO \( (\tau'(t) > -1) \), then (6)–(7) can be recovered from (9); in which case (6)–(7) and (9) has the same solutions, and (6)–(7) satisfies FIFO.

**Proof.** If \( \tau'(t) > -1 \), then (7) can be derived independently of (9), as in the Appendix. Also, rearranging (9) gives (8), and using (7) to substitute \( v(t + \tau(t)) \) for the fraction term in (8) gives (6). Thus, (9) implies (6)–(7) and, conversely, (9) was derived from (6)–(7), hence (9) and (6)–(7) are equivalent. □

To prove that (9) ensures FIFO (i.e., that \( \tau'(t) > -1 \)) we proceed as follows. We first show that in any solution of (9), \( \tau'(t) \) is continuous in \( t \) (Proposition 3). We then show (Propositions 4 and 5) that \( \tau'(t) \) can take values only in two ranges, namely a range \(-1 < \tau'(t) < M_2 < +\infty \), where FIFO is satisfied, and a negative range \( \tau'(t) < -1/\beta < -1 \), where FIFO is violated. Using these properties (continuity and boundedness of \( \tau'(t) \)) we show (Proposition 6) that if \( \tau'(t) \) starts in the first range (satisfying FIFO) at the initial time, it must remain in that range for all time thereafter. It can not jump to the FIFO, violating range because that would violate continuity. The argument is of course a bit more complicated than above. In particular, to show continuity and boundedness (in Propositions 3 and 4, respectively), we require that a certain strong condition hold (the denominator in (9) be nonzero). However, in the main FIFO proposition (Proposition 6) we find that this strong assumption (hence, continuity and boundedness) only has to be assumed at the initial point in time. If it holds at the initial time, then we are able to show, in Proposition 6, that it continues to hold for all time. This argument by continuation from initial conditions is analogous to the methods used in proving existence and uniqueness of solutions of first-order ordinary differential equation.

In the propositions below we assume:

**Assumption 1.** \( f(w) \) is continuous and strictly increasing and maps \( w \geq 0 \) onto \( \tau = f(w), 0 \leq \tau < +\infty \).

**Assumption 2.** \( u(t) \) is nonnegative and continuous in \( t \) for all \( 0 \leq t \leq T \).

Note that: Assumption (A1) implies \( f^{-1}(\tau) \) is nonnegative, continuous, and increasing on \( \tau \geq 0 \).
In Assumption (A2), continuity of \( u(t) \) for all \( 0 \leq t \leq T \) implies there is an \( M < +\infty \), such that \( u(t) \leq M \) for all \( 0 \leq t \leq T \).

In Propositions 4 and 6 we also assume that the gradients of \( u(t) \) and \( f(w) \) are bounded above.

In Propositions 5 and 6 we also assume \( u(t) \geq y > 0 \), where \( y \) may be arbitrarily small.

The above assumptions concerning \( u(t) \) and \( f(\cdot) \) are sufficient for the following propositions. It may be that slightly weaker assumptions (with alternative methods of proof) could be used to obtain similar results, for example, allowing discontinuities in \( u(t) \). The reason we have not slightly weakened the above assumptions is that our proof of FIFO (via Propositions 3–6) relies on arguments from continuity, and by weakening A1 or A2 we would lose that continuity. However, the above assumptions impose no significant restrictions on real traffic because they are observationally negligible.

A discontinuity (jump up or down) in inflow can be made continuous (smoothed) by making arbitrarily small changes in the profile of \( u(t) \), and such a change is too small to be observable for real traffic. The same applies to discontinuities in \( f(\cdot) \). Further, in numerical work, the inflow \( u(t) \) and travel time at each time \( t \) is computed to only a finite number of decimals, so there is no computational distinction between continuous and discontinuous variables. Similar comments apply to the assumption that \( f(\cdot) \) is strictly increasing, rather than just nondecreasing: We assume it is increasing to ensure that it is invertable. Again, the distinction is observationally negligible.

The results of the following three propositions (3 to 5) are needed in Proposition 6.

**Proposition 3.** Let \( 0 \leq \beta \leq 1 \), and let A1 and A2 hold. Then:

(a) The right-hand side of (9) can be written as \( \phi(\tau, t) \) and is continuous in its arguments \( \tau \) and \( t \) taken separately, when \( f^{-1}(\tau) - \beta u(t) \neq 0 \).

(b) If (9) has a solution \( \tau = \varphi(t) \) satisfying \( f^{-1}(\tau) - \beta u(t) \neq 0 \), then the solution \( \tau = \varphi(t) \) is continuous in \( t \) and has a continuous derivative \( \tau'(t) = d\varphi(t)/dt \).

**Proof.** It is well known that if any first-order ordinary differential equation \( \tau'(t) = \phi(\tau, t) \) is continuous in its arguments \( \tau \) and \( t \) taken separately, then its solution \( \tau = \varphi(t) \) is continuous in \( t \). We follow the usual line of proof.

(a) First consider the right-hand side of (9) as a function only of \( \tau \). In that case, \( u(t) \) is constant and, by A1, \( f^{-1}(\tau) \) is continuous in \( \tau \), hence the numerator and denominator are each continuous in \( \tau \). This ensures the quotient is continuous when the denominator is nonzero. Now consider the right-hand side of (9) as a function only of \( t \). Then \( f^{-1}(\tau) \) is constant and, because \( u(t) \) is continuous in \( t \) (A2), the numerator and denominator are continuous in \( t \), hence the quotient term is continuous in \( t \) because its denominator is nonzero.

(b) Let \( \tau = \varphi(t) \) be any solution of (9), hence \( d\varphi(t)/dt = \tau'(t) \) is given by (9). Because \( \varphi(t) \) is differentiable, it is continuous. The right-hand side of (9) can be written as \( \phi(\tau, t) \), and Part (a) of the proposition showed that \( \phi(\tau, t) \) is continuous in \( \tau \) and \( t \) separately. Using \( \tau = \varphi(t) \) to substitute for \( \tau \) gives \( \phi(\varphi(t), t) \), which is continuous in \( t \) because a continuous function of a continuous function is continuous. □

We now show (in Propositions 4 and 5) that \( \tau'(t) \) from (9) can take values only in certain ranges (a range that satisfies FIFO and a range that does not). We then show (in Proposition 6) that, because of continuity, \( \tau(t) \) can not escape from the range within which it starts, that is, the range containing the initial point \( (\tau, t) = (\tau_0, t_0) \). We now derive the bounds of these ranges. First define upper bounds on the gradients of \( u(t) \) and \( f(w) \), thus,

\[
\text{Assumption 3. Let } du(t)/dt \leq M_2 \text{ for all } 0 \leq t \leq T, \quad df(w)/dw \leq M_3 \text{ for all } w \geq 0, \text{ where } M_2 \text{ and } M_3 \text{ are arbitrarily large positive numbers.}
\]

This is used in Propositions 4 and 6.

**Proposition 4 (Upper Bound on \( \tau'(t) \)).** Let \( 0 \leq \beta < 1 \); let A1, A2, and A3 hold, and let \( u(t) \geq y > 0 \), where \( y > 0 \) may be arbitrarily small. Then when \( f^{-1}(\tau) - \beta u(t) \neq 0 \), we have

\[
\tau'(t) \leq [M_2 M_3 M_4/y] = M_5 < +\infty, \quad (11)
\]

where \( M_4 \) is the implicit maximum value of \( w \) noted in the proof, and assuming that (11) holds at the initial time \( t = 0 \).
Remarks. Note that the assumption \( df(w)/dw \leq M_5 \) for all \( w \geq 0 \) implies that \( f(w) \) is defined for all \( \infty > w \geq 0 \), so that we do not assume a capacity limit on flow \( w \). However, in the last paragraph of the proof we note that, in any solution of (9), \( w \) will have a finite upper limit.

We assumed that (11) holds at the initial time \( t = 0 \). This is easy to ensure. The numbers \( M_2 \) and \( M_5 \) can be chosen to be arbitrarily large, so that \( M_5 \) is arbitrarily large. From (9), at the initial time \( (t = 0) \) the value of \( f^{-1}(\tau) \) can easily be chosen, so that the fraction term in (9) is finite, hence less than \( M_5 \). Note that in the differential Equation (9), the initial value of \( \tau(t) \) is chosen to be arbitrarily large, so that \( M_5 \) is large. From (9), at the initial time (9) of \( \tau(t) \) is less than \( M_5 \). Note that in the differential Equation (9), the initial value of \( \tau(t) \) is exogenous, so that the initial value of \( f^{-1}(\tau) \) is exogenous.

Proof. It is convenient here to divide the numerator and denominator in (9) by \( u(t) \) to give

\[
\tau'(t) = -1 + (1 - \beta) [f^{-1}(\tau)/u(t) - \beta]. \tag{9'}
\]

The assumptions ensure that Proposition 3 holds, so that \( \tau(t) \) and \( \tau'(t) \) are continuous. By construction, the upper bound \( M_2 \) in (11) is positive, hence if the fraction term \( (1 - \beta)/[f^{-1}(\tau)/u(t) - \beta] \) in (9) is negative, then (9) will automatically satisfy the upper bound (11). Hence we need consider only positive fraction terms in (9'). We have assumed that \( \tau(t) \) is initially less than the proposed bound \( M_2 \) in (11), hence if \( \tau(t) \) is to move towards the upper bound \( M_2 \) in (11), it will have to increase. Because the numerator in (9') is constant, it follows from (9') that \( \tau'(t) \) can increase from its initial value only if the denominator in (9') decreases, that is, only if the first derivative of the denominator is negative. Letting the derivative of the denominator with respect to \( t \) be negative and rearranging gives \( u(t)[df^{-1}(\tau)/d\tau][\tau'(t)] < f^{-1}(\tau)[du(t)/dt] \). Because \( f(\cdot) \) is assumed strictly increasing and differentiable, \( df^{-1}(\tau)/d\tau = 1/[df(w)/dw] \). Substituting this in the preceding inequality gives

\[
\tau'(t) < f^{-1}(\tau)[df(w)/dw][du(t)/dt] / u(t). \tag{10}
\]

Because \( \tau(t) \) is continuous in \( 0 \leq t \leq T \), it is bounded above in \( 0 \leq t \leq T \) and, by A1, \( f^{-1}(\tau) \) is continuous, hence \( w(t) = f^{-1}(\tau(t)) \) is bounded above on \( 0 \leq t \leq T \). Let an upper bound on \( f^{-1}(\tau(t)) \) be \( M_4 \). Substituting this and \( du(t)/dt \leq M_2 \), \( df(w)/dw \leq M_5 \), \( u(t) \geq \mu > 0 \) into (10) gives (11). \( \square \)

Proposition 5 (Infeasible negative range for \( \tau'(t) \)). Let \( 0 \leq \beta < 1 \); let A1 hold, and \( u(t) \geq \mu > 0 \) for all \( 0 \leq t \leq T \), where \( \mu \) may be arbitrarily small. Then in any solution of (9):

(a) \( \tau'(t) \neq -1 \).

(b) \( \tau'(t) \) does not exist in the negative range \([-1/\beta, -1] \).

Remark. In (b), note that \(-1/\beta < -1 \) because \( 0 \leq \beta < 1 \), and \( \tau'(t) \) may exist in the range \(-1 < -1/\beta \).

Proof.

(a) Taking a common denominator on the right-hand side of (9) gives

\[
\tau'(t) = -[f^{-1}(\tau) - u(t)])/[f^{-1}(\tau) - \beta u(t)],
\]

which reduces to \( \tau'(t) = -1 \) if and only if \( (\beta = 1 \) or \( u(t) = 0 \)). However neither is permitted, hence \( \tau'(t) \neq -1 \).

(b) Recall that A1 implies \( f^{-1}(\tau) \) is nonnegative for all \( \tau \geq 0 \). If \( \tau'(t) < -1 \), then in (9) the fraction term must be negative, hence (multiplying this term by \(-1 \) the absolute value of the fraction term is \((1 - \beta)u(t)/[\beta u(t) - f^{-1}(\tau)] \). Then, because \( f^{-1}(\tau) \geq 0 \) and \( u(t) > 0 \), the fraction term has its smallest (absolute) value when \( f^{-1}(\tau) = 0 \), which reduces the fraction term in (9) to \(-1 - (1 - \beta)/\beta \). Therefore, the right-hand side of (9) can not take any value between \(-1 \) and \(-1 - (1 - \beta)/\beta \). Also, \(-1 - (1 - \beta)/\beta \neq -1/\beta \). \( \square \)

Proposition 6 (Upper and lower bounds on \( \tau'(t) \) for all time \( t \), \( 0 \leq t \leq T \)). Let \( 0 \leq \beta < 1 \) and A1, A2, and A3 hold. At the initial point \( (t, \tau) = (0, \tau_0) \), let \([f^{-1}(\tau(t)) - \beta u(t)] > 0 \), so that (9) implies \( \tau'(t) > -1 \), at the initial time \( t = 0 \).

Then, in any solution of (9), \(-1 < \tau'(t) < M_5 < +\infty \) for all time \( t \), \( 0 \leq t \leq T \), where \( M_5 \) is defined in Proposition 4.

Remark. We assume \([f^{-1}(\tau(t)) - \beta u(t)] > 0 \) only at the initial time \( t = 0 \). We then prove it for all later time. Note from (9) that assuming \([f^{-1}(\tau(t)) - \beta u(t)] > 0 \) at the initial time \( t = 0 \) is equivalent to assuming \( \tau'(t) > -1 \), that is, FIFO, at the initial time \( t = 0 \).

Proof. The assumptions ensure that Proposition 5 holds, so \( \tau'(t) \) does not exist in the negative range \([-1/\beta, -1] \). The assumptions ensure that:
At the initial time $t = 0$, Proposition 4 holds, so that in any solution of (9), $\tau(t) < M_5 < +\infty$.

At the initial time $t = 0$, $\tau(t) > -1$ (from substituting $[f^{-1}(\tau(t)) - \beta u(t)] > 0$ in (9)).

Hence, initially $\tau'(t)$ lies in the range $-1 < \tau'(t) < M_5 < +\infty$. Also, the assumption that $[f^{-1}(\tau(t)) - \beta u(t)] > 0$ at the initial time means that Propositions 3 holds at the initial time, therefore $\tau(t)$ is initially continuous. It is continuous so long as $[f^{-1}(\tau(t)) - \beta u(t)] \neq 0$. However $[f^{-1}(\tau(t)) - \beta u(t)] = 0$ implies, from (9), that $\tau'(t) = +\infty$, hence to attain $[f^{-1}(\tau(t)) - \beta u(t)] = 0$, $\tau'(t)$ would have to jump over the upper bound $M_5$ obtained in Proposition 4, which would contradict continuity of $\tau'(t)$. We can apply this argument continuously over time. At each point in time, $\tau'(t)$ is continuous and is in the range $-1 < \tau'(t) < M_5 < +\infty$. To escape from this range it would have to jump over the upper bound $\tau'(t) < M_5 < +\infty$ or the lower bound $\tau'(t) > -1$ (obtained in Propositions 4 and 5, respectively), which would contradict continuity of $\tau'(t)$. Therefore $\tau'(t)$ has to remain in the range $-1 < \tau'(t) < M_5 < +\infty$ for all time $t$. $\square$

**Corollary 1** (9) satisfies FIFO. Let the assumptions in Proposition 6 hold. Then, if (9) satisfies FIFO (i.e., $\tau'(t) > -1$) at the initial time $t = 0$, it satisfies FIFO for all time $t$, $0 \leq t \leq T$.

**Proof.** The proof follows immediately from Proposition 6. $\square$

**Corollary 2 (Model (6)–(7) satisfies FIFO).** Let the assumptions in Proposition 6 hold. Then, if (6)–(7) satisfies FIFO (i.e., $\tau'(t) > -1$) at the initial time $t = 0$, it satisfies FIFO for all time $t$, $0 \leq t \leq T$.

**Proof.** The proof follows immediately from Corollary 1 and Proposition 2. $\square$

**Remark.** In the above propositions the assumption $\beta < 1$ rules out $\beta = 1$. If we let $\beta = 1$, then (6) and (8) both reduce to $\tau(t) = f(u(t))$, and (9) is not obtainable. It is well known that if $\tau(t) = f(u(t))$ and inflows are falling off sufficiently rapidly over time, then FIFO can be violated. (More formally, $\tau(t) = f(u(t))$ implies $\tau'(t) = [df(u)/du]du/dt$ evaluated at $u = u(t)$, therefore if $du/dt$ is negative and $|du/dt| > 1/(df(u)/du)$ evaluated at $u = u(t)$, then $\tau'(t) < -1$. Thus, if $\beta = 1$, then the model (6)–(7) does not satisfy FIFO, for some patterns of inflows.

Therefore it, seems very surprising that if $\beta$ is “almost” equal to 1 (i.e., $\beta = 1 - \epsilon$ where $\epsilon$ is arbitrarily small), then (6)–(7), or equivalently (8) or (9), satisfies FIFO for the same pattern of inflows. In other words, if the argument of $f(\cdot)$ in (6) is $u(t)$, it does not satisfy FIFO, but if its argument is say $0.999u(t) + 0.001\tau(t + \tau(t))$, it is sufficient to ensure that it satisfies FIFO. In effect, if we take even the slightest account of outflows, as well as inflows, when computing the link travel times, this will ensure FIFO!

2. **Solving the Link Travel-Time Model (8), or (6)–(7) and Comparisons with Other Models**

2.1. **Algebraic Solutions**

The link travel-time model (8) is a first-order differential Equation (9). For most forms of $f(\cdot)$ and inflow patterns $u(t)$ it will be difficult or impossible to solve for $\tau$ as an explicit function of $t$. To illustrate the difficulty, we consider finding a solution for one very simple case, and it will normally be more difficult for more general cases. In view of that, in later sections instead solve using numerical methods.

**Example 1.** See Figure 1, with a linear $f(\cdot)$ and step function inflows.

Suppose that $f(\cdot)$ in (6) and (8) is of linear form $f(w) = a + bw$. Using this in (8) and simplifying reduces (8) to

$$\tau(t) = a + bu(t) \left( \frac{1 + \beta \tau(t)}{1 + \tau(t)} \right). \quad (12)$$

Rearranging to make $\tau(t)$ the subject gives

$$\tau'(t) = - \frac{\tau(t) - a - bu(t)}{\tau(t) - a - b \beta u(t)}. \quad (13)$$

To solve this, assume constant inflow $u(t) = U$, from let us say time $t = t_0$. From (13) we can write $d\tau = \cdot dt$, hence $d\tau/\cdot = dt$, and integrating this gives $(\beta - 1)bU \ln(\tau - a - bU) - \tau + C = t$. To determine the constant $C$, let $\tau = \tau_0$ at an initial time $t = t_0$. Using this value of $C$ gives

$$t = -(1 - \beta)bU \ln \left( \frac{a + bU - \tau}{a + bU - \tau_0} \right) - \tau + (\tau_0 + t_0). \quad (14)$$
For example, replacing the derivative in (9) using one of the various methods for solving first-order differential equations (e.g., Burden and Faires 1993). For example, replacing the derivative in (9) with a forward difference gives

$$\frac{\tau(t + \Delta t) - \tau(t)}{\Delta t} = \frac{-f^{-1}(\tau) - u(t)}{f^{-1}(\tau) - \beta u(t)}, \quad (15)$$

hence

$$\tau(t + \Delta t) = \tau(t) - \frac{f^{-1}(\tau) - u(t)}{f^{-1}(\tau) - \beta u(t)} \Delta t, \quad (16)$$

and we can use this to compute \( \tau(t) \) sequentially at times \( t = \Delta t, 2\Delta t, \) and so on, if given \( \tau(t) \) at time \( t = 0 \) and \( u(t) \) at times \( t = 0, \Delta t, 2\Delta t, \ldots \). We also need \( f^{-1}(\cdot) \), which may be given analytically or computed numerically from a given \( \tau = f(w) \), which itself may be in analytic form or interpolated or estimated from a table of values. To obtain the initial travel time \( \tau(0) \), we can start at a time of day when the inflow \( u(0) \) yields a free-flow travel-time.

Alternatively, if we initially know or can estimate the value of \( \tau(t) \) at \( t = T \), then we can use backward differencing. For example, use (16), with \( \tau(t + \Delta t) \) on the left-hand side replaced by \( \tau(t - \Delta t) \), to compute \( \tau(t) \) at times \( t = T - \Delta t, T - 2\Delta t, \) and so on.

### 2.3. Numerical Examples

In this section we present some illustrative numerical examples, based on applying the model (8) to various simple inflow profiles. To further illustrate the results, we also present the corresponding solutions from the model (1) and from a deterministic point-queue model. These seem to be the most popular alternative whole-link delay models used in dynamic traffic assignment.

Comparing with the Friesz et al. (1993) Linear Model. Equation (8) states the travel time as a function of a (weighted average) flow \( w \) given in (5), while (1) states it as a function of \( x \), the number of vehicles on the link. To make any meaningful comparisons of the numerical results from models (8) and (1), we need to use functional forms and parameters in (8) that are comparable with those in (1). To make the models comparable, we choose the form of \( f(\cdot) \) in (8), so that (8) and (1) yield the same travel times and outflows when these are constant over time. When travel times and flows are constant, (1) reduces to

\[\tau = \frac{a + bU - \beta bU}{(a + bU - \tau_0)} \to \alpha \quad \text{as} \quad \tau \to \infty.\]

(If the inflows prior to time \( t = t_0 \) may be variable but are not needed for the solution (14): We need know prior to the initial travel time \( \tau = \tau_0 \).) We cannot invert (14) to make \( \tau \) the subject. Note, however, that \( t \to \infty \) as \( \ln(a + bU - \tau)/(a + bU - \tau_0) \to -\infty \). That is, \( t \to \infty \) as \( (a + bU - \tau)/(a + bU - \tau_0) \to 0 \) or as \( \tau \to a + bU \). This is what we would expect: For any starting value \( \tau = \tau_0 \), the link travel time converges asymptotically to the level \( \tau = a + bU \) implied by a constant flow rate \( U \).

Though we can not invert (14) we can easily illustrate it graphically. Let \( t_0 = 0, a = 0.2, b = 0.125, U = 4 \), and graph (14) for three different initial values of \( \tau \), namely, \( \tau_0 = (a + bU) - \beta bU \), \( \tau_0 = (a + bU) + \beta bU/2 \), and \( \tau_0 = (a + bU) + \beta bU \). The asymptote is then \( \tau = a + bU = 0.7 \), and the three initial values of \( \tau \) are respectively 0.25 below the asymptote and 0.125 and 0.25 above it. We observe that the resulting travel times \( \tau(t) \), in Figure 1, converge smoothly to the asymptote whether they start above or below it.

### 2.2. Numerical Solutions

As noted above, (9) or (8) can be solved numerically using one of the various methods for solving first-order differential equations (e.g., Burden and Faires 1993). For example, replacing the derivative in (9)
\[ \tau = a + bx, \] and we also have the identity \[ x = \omega \tau. \] Substituting \( x = \omega \tau \) in (1) and rearranging this gives the travel time as a function of \( \omega \), thus

\[ \tau = a/(1 - bw). \] (17)

We use this as the functional form for \( f(\cdot) \) in (8) and (6). By replacing \( \omega \) with the “average” flow from (5), (17) becomes

\[ \tau(t) = \frac{a}{1 - b(\beta u(t) + (1 - \beta)v(t + \tau(t)))}. \] (18)

When \( \omega = 0 \), (17) reduces to the free flow travel time \( \tau \) and, as \( \omega \to 1/b \) from below, \( \tau \to +\infty \), hence (17) implies that the maximum flow capacity for the link is \( 1/b \). In (18) we can, in principle, have \( u(t) \) greater than \( 1/b \) as long as \( \beta u(t) + (1 - \beta)v(t + \tau(t)) \) is less than \( 1/b \). However, because the travel-time model (17) implies a flow capacity of \( 1/b \), allowing inflows greater than that does not seem to be a valid use of the model. For model (1) the maximum feasible level for constant flow, with inflow equal to outflow \( (u(t) = v(t)) \), is also \( 1/b \) because in that case \( x = \omega \tau \), which reduces (1) to (17).

**Comparing with a Point-Queue Model.** One interpretation of the model (1) is that it represents a fixed travel time “\( a \)” on the link plus a congestion-related queuing time \( bx \) based on a service rate \( 1/b \) (Friesz et al. 1993, p. 185). Daganzo (1995) gives the same interpretation of (1) and calls \( \tau = bx \) a point-queue model.

In some network models, congestion is modeled as deterministic point-queues rather than as congestion while travelling along links. That is, assume a constant travel time \( a \) on the link and a point queue at the beginning or end of the link with queue outflow capacity \( 1/b \). Then if there is a queue (i.e., \( x(t) > 0 \)), outflow capacity will be at its capacity level \( v(t) = 1/b \) and traffic entering at time \( t \) will exit at time \( \tau(t) = x(t)/(1/b) = bx(t) \). If there is no queue (i.e., \( x(t) = 0 \)), then inflow exits immediately; that is, \( v(t) = u(t) \) and \( \tau(t) = 0 \). The total travel delay with this model is thus

\[ a + \tau(t) \] where \( \tau(t) \) is the dwell time in the deterministic queue. (19)

It is interesting to compare this deterministic queue model with (18) and (1). In comparing (18) and (1) above (and in the numerical examples below) we assume that the inflow to the link does not exceed the link-flow capacity \( 1/b \). However, if we also assume this for the deterministic queue model (19), there will be no queue. In that case the travel times for the three models are given by (18), (1), and \( a \), respectively.

Though we assumed that for models (18) and (1) the arrival rate at the link does not exceed capacity, in practice this may not be true. To handle that, we can assume a deterministic point just before the link, with an outflow capacity \( 1/b \), so that (18) is replaced by

\[ (18) \] with the link preceded by a deterministic point-queue with outflow capacity \( 1/b \). (20)

This approach (i.e., preceding the travel link with a point queue) has been adopted in other network models, for example, with the cell-transmission model, to ensure link flows do not exceed capacity. Comparing (20) with (19), we note that the queue size and dwell times are the same in each model. Hence, even with a point-queue inserted before the link, the difference between the three models is again that the travel times are given by (18), (1), and \( a \), respectively.

Below we set out four numerical examples. These are primarily to illustrate model (18), but to help put the examples in context we then compare with model (1) and with the assumption of a free-flow travel time. In all of the examples we have assumed that the arrival rate at the link is less than the capacity \( 1/b \). That means that we do not need to introduce a point-queue to restrict inflow to the link. Also, the deterministic queue model (19) reduces to simply a travel time \( a \), with outflows equal to inflows shifted forward in time by \( a \). (The examples below are not directly comparable with Figure 1 because for Figure 1 we assumed a travel-time function of the form \( f(\omega) = a + bw \), while in the examples below we assume the form (17) hence (18).)

In Examples 2 to 4, we assume the parameters of (17) and (1) are \( a = 10, b = 0.125 \), hence a link capacity \( 1/b = 8 \). We also assume \( \beta = 0.5 \). In Example 5 we
assume a linear travel-time function (6), that is, \( \tau(t) = 20 + 0.75w \), where \( w \) is given by (5).

**Example 2 (Figure 2).** In this example we assume a simple inflow profile that could approximately represent a morning or evening peak (see Figure 2). Traffic builds up from zero to a peak, remains there for some time, and falls off again to zero. The build-up and decline is “relatively” slow because the time taken to build up from zero to near-peak inflow is about five times the free-flow travel time \( a = 10 \). The peak inflow rate \( u(t) \) is set to \( U = 4 \), which is less than the link outflow capacity \( 1/b = 8 \). The inflow function is

\[
u(t) = \begin{cases} 
4 \sin \left( \frac{\pi t}{100} \right) & 0 \leq t < 50 \\
4 & 50 \leq t < 120 \\
4 \sin^5 \left( \frac{\pi t}{240} \right) & 120 \leq t < 240.
\end{cases}
\]

The resulting profiles of link travel times and outflow are quite similar for models (18) and (1). For both models, the outflows build up to equal the constant peak-inflow rate and decline again, and travel times build up to the level \( \tau = 20 \), corresponding to constant inflows \( U = 4 \).

There is a feature of the outflow profile for model (18) that is worth noting. When the inflow begins to decline quickly, outflow at first increases a bit before declining. This “bump” in outflow is caused as follows. At about time 120, the inflow starts to decline quickly from the previous constant level. This causes travel times to decline quickly, and if travel times decline sufficiently quickly, the traffic entering in one unit of time may exit over a shorter span of time. This implies an outflow rate greater than the inflow rate, hence the “bump” up in the outflow. The bump in outflow starts a certain time after the inflow starts to fall, the delay being due to the time taken for the flow to traverse the link.

**Example 3 (Figure 3).** In Example 2 we assumed that inflows increase relatively “slowly” from zero to a peak. Here, as a more severe test of the models, we let inflows increase very rapidly, growing from zero to near their peak value \( U = 6 \) in the free-flow travel time \( a = 10 \) for the link (see Figure 3). The inflow function is

\[
u(t) = \begin{cases} 
6 \sin \left( \frac{\pi t}{20} \right) & 0 \leq t < 10 \\
6 & 10 \leq t < 120 \\
6 \sin^5 \left( \frac{\pi t}{240} \right) & 120 \leq t < 240.
\end{cases}
\]

The resulting outflows and travel times for all three models are shown in Figure 3. For model (18), the outflows increase smoothly from zero to \( U \), and for
model (1) they increase a bit less smoothly, but are otherwise surprisingly similar. For both models, the travel times converge to \( \tau = 40 \), which (from (17)) corresponds to a constant inflow \( U = 6 \).

**Example 4** (Figure 4). In Examples 2 and 3 the inflows change smoothly over time. Also, in deriving sufficient conditions for FIFO for model (8), we assumed for convenience that inflows changed smoothly (were continuous and differentiable). In this example we test the model by applying it to discontinuous changes in inflows. Assume there are initially no vehicles on the link and the inflow is zero. Let the inflows then jump to a fixed value \( U = 4 \) and remain there indefinitely, thus

\[
u(t) = \begin{cases} 0 & t < 40 \\ 4 & t \geq 40 \end{cases}
\]

The value \( U = 4 \) is set less than the link-flow capacity \( 1/b = 8 \), otherwise in all three models the link travel time would simply grow over time to infinity. In Figure 4(b) the travel times converge to the travel time corresponding to constant inflow \( U \) (i.e., \( \tau = 20 \)). For model (18) the outflows increase smoothly from zero to \( U \). For model (1) the outflows converge to \( U \) in
a series of decreasing size steps. This step behavior was examined and explained in Carey and McCartney (2000, 2002). Note that this is not a significant criticism of the model (1) because whole-link models are not intended to accurately handle sudden or very rapid changes in flows.

Example 5 (Figure 5). One of the main purposes of this paper is to show that the model (6)–(7), hence (8), ensures FIFO. The present example is designed to pose a challenge to FIFO. We noted in the introduction that FIFO is violated only if travel times decline more rapidly than $\tau'(t) > -1$. To generate a rapid decline in travel times we here generate a rapid decline in inflows. In Figure 5 we assume the fastest possible decline in inflow, namely a discontinuous fall. The resulting travel times $\tau(t)$ and $\tau'(t)$, computed from model (18), are shown in Figure 5. The large, sharp fall in inflow causes $\tau'(t)$ to jump down to almost $-1$, but it never achieves $\tau(t) = -1$, nor does it violate $\tau'(t) > -1$. In this example we deliberately choose inflows that decline even faster than is allowed in the proofs of FIFO in §1. There, when proving FIFO we assumed for convenience that inflow was continuous and its gradient was bounded. Though the inflows in Figure 5 do not satisfy these sufficient conditions, FIFO holds. Incidentally, by making an arbitrarily small change to the inflow profile in Figure 5 we can convert the step-function inflows in Figure 5 into continuous inflows that satisfy the assumptions in the FIFO proofs. However, because the computations are performed by a discrete method, using very small time steps, there is no computational difference between a continuous and a discontinuous function.

The above examples illustrate various interesting features of the model (6)–(7), and also illustrate that these differ in expected or explainable ways from the behavior of model (1). However, the differences in the model predictions are not dramatic. We could generate examples in which the differences are greater. However, that would not, and could not, determine or demonstrate which model is “best.” The purpose of the present paper is not to compare these two whole-link models, but to introduce the new model and show that it has desirable properties, similar to those of the existing model $\tau(t) = f(x)$. In later work we will compare the model more exhaustively with other models including nonwhole-link models that treat traffic flow, speed, and density varying along the link.

3. Concluding Remarks
When traffic flows are varying over time, so called whole-link models have been used to model or approximate the travel times on individual links. This is especially common in network models for dynamic traffic assignment. A whole-link model that has been
much discussed and used in recent years is one that treats the link travel time as a function of the number of vehicles on the link. In particular, it has been shown that, under mild assumptions, that model satisfies FIFO.

We propose an alternative whole-link travel-time model. For a vehicle entering a link at time \( t \), we let the link travel time be a function of the “average” flow rate that the vehicle experiences on the link. As an estimate of the average flow rate we use a weighted average of the inflow rate at the time the vehicle enters the link and the outflow rate at the time it exits. We show that this travel-time model ensures FIFO and has other desirable properties. A somewhat surprising result is that if, in the weighted average, we give even an arbitrarily small weight to the outflow rate, then FIFO is ensured, but if we give no weight to the outflow rate, then FIFO can be violated.

In §2 we present numerical examples to test and illustrate the behavior of the model and find these consistent with its theoretical properties. In particular, we generate an example to try to force a FIFO violation and show the model stops short of violating FIFO. We also find that, in these numerical examples, the outflows obtained by the new model and the Friesz et al. (1993) model are fairly similar, which is not surprising. The latter model treats travel time as a linear function of the number of vehicles on the link.

A discrete-time version the travel-time model (8) introduced here has also been applied to modeling the travel times on the individual links in Carey (1999, 2001). Carey (1999) develops a network model for dynamic traffic assignment, in which the travel times on the individual links can be modeled using any of various travel-time models. In a series of numerical experiments in that paper, a discrete version of (8) was used (with \( \beta = 0.5 \)) to model the individual links. However, the model (8) is not set out or discussed in Carey (1999), and none of the discussion or results in the present paper are presented there.

Appendix

The following conditions for first-in-first-out (FIFO) and traffic conservation are well known in the literature and are used at various points in this paper. They are derived here only for ease of reference.

**Derivation of** \( \tau(t) > -1 \).

Let time be divided into discrete periods or intervals \( t = 1, \ldots, T \), each of length \( \Delta t \). Traffic entering the link at time \( t \) exits at time \( t + \tau(t) \), and traffic entering at time \( t + \Delta t \) exits at time \( (t + \Delta t) + \tau(t + \Delta t) \). Hence, for traffic to exit in the same order that it entered (i.e., FIFO), it is necessary that \( (t + \Delta t) + \tau(t + \Delta t) > t + \tau(t) \), hence \( \Delta t + \tau(t + \Delta t) - \tau(t) > 0 \). Dividing through by \( \Delta t \), letting \( \Delta t \to 0 \), and assuming that \( \tau(t) \) is differentiable gives

\[
\tau(t) > -1, \tag{A.1}\]

which is a well-known, necessary, and sufficient condition for FIFO when time is treated as continuous.

**Derivation of Equation (7).** If we assume that traffic is conserved along the link (is not dissipated or generated along the link) and exits in the same order that it entered (i.e., FIFO), then traffic entering up to time \( t \) will exit up to time \( t + \tau(t) \) where \( \tau(t) \) is the link travel time for traffic entering at time \( t \). That is,

\[
\int_0^t u(s) \, ds = \int_{t-\tau(t)}^{t} v(s) \, ds \tag{A.2}\]

where \( u(s) \) and \( v(s) \) are the inflow and outflow rates, respectively, at time \( s \). Differentiating (A.2) with respect to \( t \) gives

\[
u(t) = v(t + \tau(t)) (1 + \tau'(t)), \tag{A.3}\]

and rearranging (A.3) gives (7).

References


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