Comparing whole-link travel time models

Malachy Carey *, Y.E. Ge

Faculty of Business and Management, University of Ulster, Newtonabbey, Northern Ireland BT37 0QB, UK

Received 25 October 2001; received in revised form 3 September 2002; accepted 23 September 2002

Abstract

In a model commonly used in dynamic traffic assignment the link travel time for a vehicle entering a link at time $t$ is taken as a function of the number of vehicles on the link at time $t$. In an alternative recently introduced model, the travel time for a vehicle entering a link at time $t$ is taken as a function of an estimate of the flow in the immediate neighbourhood of the vehicle, averaged over the time the vehicle is traversing the link. Here we compare the solutions obtained from these two models when applied to various inflow profiles. We also divide the link into segments, apply each model sequentially to the segments and again compare the results. As the number of segments is increased, the discretisation refined to the continuous limit, the solutions from the two models converge to the same solution, which is the solution of the Lighthill, Whitham, Richards (LWR) model for traffic flow. We illustrate the results for different travel time functions and patterns of inflows to the link. In the numerical examples the solutions from the second of the two models are closer to the limit solutions. We also show that the models converge even when the link segments are not homogeneous, and introduce a correction scheme in the second model to compensate for an approximation error, hence improving the approximation to the LWR model.

© 2003 Elsevier Ltd. All rights reserved.

1. Introduction

In several papers concerning dynamic traffic assignment (DTA), the travel time on a link is treated as a function of the number of vehicles on the link. That is, $\tau(t) = f(x(t))$, where $\tau(t)$ is the link travel time for vehicles entering a link at time $t$ and $x(t)$ is the number of vehicles on the link at time $t$. This model was introduced in network models for DTA by Friesz et al. (1993), and has been used, and its properties, in particular a first-in-first-out (FIFO) property, have been investigated extensively, by Friesz et al. (1993), Astarita (1995, 1996), Wu et al. (1995, 1998), Xu et al.
(1999), Adamo et al. (1999), Chabini and Kachani (1999), Zhu and Marcotte (2000), Friesz et al. (2001) and Carey and McCartney (2002). An equivalent form, as a function of link density, is used in Jayakrishnan et al. (1995) and Ran et al. (2002). The function $\tau(t) = f(x(t))$ is also a special case of a travel time function used for example in Ran et al. (1993), Ran and Boyce (1996) and Boyce et al. (2001): these papers include inflows or outflows as arguments of the travel time function in addition to $x(t)$.

More recently Carey et al. (2003) introduced an alternative travel time model in which the travel time for a user is taken as a function of $w(t)$, an estimate of the average flow rate in the neighbourhood of the user while traversing the link. More formally, they write the travel time for a user entering at time $t$ as $\tau(t) = h(w(t))$, and let $w(t)$ be a weighted average of the inflow rate $u(t)$ when the user enters the link and the outflow rate $v(t + \tau(t))$ when the same user is exiting from the link at time $t + \tau(t)$. A discretised version of this model was used in Carey (1999, 2001).

As a benchmark for comparing the above two travel-time models, we use the hydrodynamical Lighthill, Whitham, Richards (LWR) model (Lighthill and Whitham, 1955; Richards, 1956). We use the latter (the LWR model) since it is widely accepted both theoretically and empirically (e.g. Newell, 1989; Daganzo, 1997). In numerical examples, we use a finite difference approximation to the LWR model (Daganzo, 1995). First we compare the $\tau(t) = f(x(t))$ and $\tau(t) = h(w(t))$ models directly with the LWR model when the former are applied to the whole link. Then, to make the $\tau(t) = f(x(t))$ and $\tau(t) = h(w(t))$ models more comparable with the LWR model, we divide the link into segments and apply the models $\tau(t) = f(x(t))$ or $\tau(t) = h(w(t))$ sequentially to the segments: we of course (as in Section 3.3) must first adjust the parameters of $f(\cdot)$ and $h(\cdot)$ to take account of the segment lengths. We compare the solutions of the three models when the link discretisation for the two whole link models is far from refined, and also when the links are not discretised at all, since the latter is the usual mode in which the travel time models $\tau(t) = f(x(t))$ and $\tau(t) = h(w(t))$ are used in the literature. For a homogeneous link without queues, we examine how the solutions of the $\tau(t) = f(x(t))$ and $\tau(t) = h(w(t))$ converge to that of the LWR model as the link discretisation is refined. This convergence for the $\tau(t) = f(x(t))$ model was also discussed in Carey and Ge (2001).

Following Heydecker and Addison (1998), we refer to models such as $\tau(t) = f(x(t))$ or $\tau(t) = h(w(t))$ as ‘whole link’ models, because the variables in the model (inflows, outflows, travel times) refer to the whole link, or whole section of a link, in contrast to models such as LWR which consider variables varying continuously along the link. This raises a question of terminology: when we divide a link into segments and apply the model $\tau(t) = f(x(t))$ or $\tau(t) = h(w(t))$ (adjusted for segment lengths) to the segments, can we still refer to these as whole link models? The answer is yes, since the term ‘whole link model’ refers to the form of the model, not to the length of the link.

The discretised models $\tau(t) = f(x(t))$ and $\tau(t) = h(w(t))$ presented here divide the link into segments or cells as does the cell transmission model (Daganzo, 1994). However, the models differ in the way that they compute the movement of traffic over time. The cell transmission model computes the amount of traffic that will exit from a cell in the next time step, where the time step is less than or equal to the free flow travel time for the cell. In contrast, for the $\tau(t) = f(x(t))$ and $\tau(t) = h(w(t))$ models, when applied to a discretised link, for users entering a segment at time $t$, the model immediately computes the time $t + \tau(t)$ at which the users will exit from the segment.

The present paper should assist developers of DTA models in making more informed choices about travel time models, and about the choice of link lengths to use in such models. When
applying a DTA model to a given network, users have choices in defining numbers and lengths of links. For example, should a long link be treated as several shorter links? This choice, and its effect on the accuracy or quality of the DTA solutions, is usually not discussed but is important. When applying the travel time models to link segments, we do not change the basic form of these models, hence the theory and algorithms already developed in the existing DTA literature based on these link models should also apply when we use the discretised forms from the present paper.

As is normally assumed in the DTA literature where the travel-time model \( \tau(t) = f(x(t)) \) has been used, we assume here that \( \tau(t) = f(x(t)) \) represents delay on a link and that there are no other obstructions, bottlenecks, or controls on the link, including at the link exit, other than any that may be interpreted as being implicitly present in the form of the function \( f(\cdot) \). Also, since the function does not refer to any spatial dimension along the link, it implicitly assumes that the link is homogeneous. For these circumstances, it is shown in Carey and Ge (2002) that (if the link inflow is not allowed to exceed the capacity given by the peak of the flow-density function) then that part of the \( \tau = f(x) \) function corresponding to the downward sloping part of the flow-density function will not be attained and hence not utilised. It is well known that this is also true for the LWR model, i.e., for a homogenous link without obstructions or controls, the downward sloping part of the flow-density function is not attained hence not utilised. For the \( \tau(t) = h(w(t)) \) function introduced in Carey et al. (2003) the backward bending part of \( \tau = h(w) \) corresponds to the downward sloping part of the flow-density function. Hence (in the scenario considered above) the backward bending part of the \( \tau = h(w) \) curve is not attained or utilised: only the lower single-valued part of the \( \tau = h(w) \) curve is attained and utilised. Despite the above, we note that in some examples in the literature, when the model \( \tau(t) = f(x(t)) \) was applied to a homogenous link without obstructions or controls, the part of the travel time model \( \tau(t) = f(x(t)) \) corresponding to a downward sloping flow-density function has been attained. However, this appears to be due to the inflow to the link being allowed to exceed the link flow capacity. In view of the above paragraph, in this paper we consider only homogeneous links without obstructions or traffic controls, and assume that the link inflow is not allowed to exceed the capacity given by the peak of the flow-density function.

We compare the above models using the functional forms set out in Section 3. We use linear and quadratic forms for the travel time function \( \tau(t) = f(x(t)) \) since these are the forms most commonly used in the above DTA literature. We then derive the corresponding functional forms for the \( \tau(t) = h(w(t)) \) model. In Section 4, we apply these various forms to various inflow patterns and compare the results.

2. The two travel time models

In the next two subsections we set out more fully the two models to be compared.

2.1. The travel time model \( \tau(t) = f(x(t)) \)

As noted in the introduction, in DTA for networks, various authors have used or investigated the following model for travel time on each link:

\[
\tau(t) = f(x(t)),
\]

(1)
where $f(\cdot)$ is nondecreasing and continuous,

$$x(t) = x(0) + \int_0^t (u(s) - v(s)) \, ds$$

and $u(s)$ and $v(s)$ are the inflow and outflow rates respectively for the link at time $s$. Friesz et al. (1993) showed that when $f(x)$ is linear the model satisfies FIFO for all continuous inflow patterns $u(t)$ and later authors (e.g. Xu et al., 1999) obtain fairly weak FIFO conditions for when $f(x)$ is nonlinear. When FIFO holds and flow is conserved,

$$x(0) + \int_0^t u(s) \, ds = \int_0^{t+s(t)} v(s) \, ds$$

and differentiating w.r.t. $t$ gives

$$v(t + \tau(t)) = \frac{u(t)}{1 + \tau'(t)} = \frac{u(t)}{1 + f'(x)(u(t) - v(t))},$$

where $\tau'(t) = d\tau/dt$ and $f'(x) = df(x)/dx$. To solve the model 1,2, note that when $u(t)$, $v(t)$ and $x(t)$ are given at time $t$, (4) gives $v(t + \tau(t))$, that is, $v(t)$ for a time $\tau(t)$ ahead. Hence when solving 1,2 sequentially over time, $v(t)$ is already known at each time $t$. Then, with $v(t)$ known and $u(t)$ given, at each time $t$, from $t = 0$ to $t = T$, use (2) to compute $x(t)$ and (1) to compute $\tau(t)$.

Carey and Ge (2001) set out a discretised version of the above travel time model, dividing the link into segments and applying the model sequentially to these segments. They also allow inhomogeneity in the segments, so that the segment traversal times depend on the location of the segment and on the time $t$ at which users enter the segment. If the discretisation is refined to the continuous limit, the solution profiles (for outflows and travel times) are the same as for the LWR model.

2.2. The travel time model $\tau(t) = h(w(t))$, where $w(t)$ is the estimated mean flow rate

Carey et al. (2003) set out an alternative link travel time model as follows. Let the link travel time function be

$$\tau(t) = h(w(t)),$$

where $h(\cdot)$ is continuous, $w(t)$ is an estimate of the flow in the immediate neighbourhood of a vehicle, averaged over the time the vehicle is traversing the link. Let $u(t)$ and $v(t)$ be the inflow and outflow rates respectively for the link at time $t$. Then $u(t)$ is the inflow rate when the vehicle enters the link at time $t$ and $v(t + \tau(t))$ is the outflow rate when it is exiting from the link at time $t + \tau(t)$. An estimate of the average flow rate associated with the vehicle as it traverses the link can therefore be estimated reasonably as

$$w(t) = \beta u(t) + (1 - \beta) v(t + \tau(t)),$$

where $\beta$ is a weighting constant $1 > \beta > 0$. To define the outflow rate $v(t + \tau(t))$ they use the flow propagation equation
can be rewritten in an equivalent form, subtracting (9) from this equation gives
\[ x = \frac{u(t)}{1 + \tau'(t)}. \]

Combining (5)–(7), the model can be written as
\[ \tau(t) = h\left(\beta u(t) + (1 - \beta) \frac{u(t)}{1 + \tau'(t)}\right) \]
which is entirely in terms of variables \( u(t) \) and \( \tau'(t) \) for time \( t \), hence can be rearranged as a first
order partial differential equation \( g(\tau'(t), \tau(t), u(t)) = 0 \). Carey et al. (2003) show that the travel-
time model (8), or (5)–(7), satisfies a FIFO property of traffic flow and has other desirable
properties. Though the above is the complete model, two further equations implied by (7) are
useful below. First,
\[ x(0) + \int_0^t u(s) \, ds = \int_0^{t+\tau(t)} v(s) \, ds \]
is obtained by rearranging (7) and integrating. Second, a conservation equation at time \( t \) is not
explicitly included in the statement of the model (5)–(7), though (7) can be interpreted as an inter-
temporal conservation equation. However, it is derived as follows.

**Proposition 1.** Eq. (7), for \( 0 \leq t \leq T \), together with the FIFO property of model (5)–(7), implies a
conservation equation
\[ x(t) = x(0) + \int_0^t (u(s) - v(s)) \, ds \quad \text{for } \tau(0) \leq t \leq T. \]

**Remark.** Eq. (7) determines outflows \( v(t + \tau(t)) \) only from time \( t = \tau(0) \) onwards, hence outflows
from \( t = 0 \) to \( t = \tau(0) \) are assumed given. Also, for conservation, we must exogenously set the
initial loading at \( x(0) = \int_0^{\tau(0)} v(s) \, ds \) and, for \( t = 0 \) to \( t = \tau(0) \) set \( x(t) = x(0) + \int_0^t (u(s) - v(s)) \, ds \).

**Proof.** The FIFO property implies that all traffic on the link at time \( t + \tau(t) \) must have entered
since time \( t \). That is, \( x(t + \tau(t)) = \int_t^{t+\tau(t)} u(s) \, ds \) (this holds from \( t = 0 \) to \( t \) such that \( t + \tau(t) = T \)).
Subtracting (9) from this equation gives \( x(t + \tau(t)) = x(0) + \int_0^{t+\tau(t)} (u(s) - v(s)) \, ds \) (from \( t = 0 \) to \( t \)
such that \( t + \tau(t) = t \)). Letting \( t' \), denote \( t + \tau(t) \), this can be rewritten as \( x(t') = x(0) + \int_0^{t'} (u(s) - v(s)) \, ds \) from \( t' = \tau(0) \) to \( t' = T \). Dropping the prime gives (10). \( \square \)

2.2.1. A discretised \( \tau(t) = h(w(t)) \) model and convergence to the LWR model

We now consider how the model (5)–(7) behaves if we apply it sequentially to each segment of a
discretised link, and what happens as the segment length is reduced towards zero. In (5)–(7),
distance is not explicitly stated, being included in the form of \( h(\cdot) \). To make it explicit, let \( L \) denote
the link length and \( \bar{\tau}(t) = \tau(t)/L \) denote the travel time per unit distance \( (= 1/\text{speed}) \), so that (5)
can be rewritten in an equivalent form,
\[ \bar{\tau}(t) = h(w(t))/L = \bar{h}(w(t)). \]

\[ v(t + \tau(t)) = \frac{u(t)}{1 + \tau'(t)}. \]
We can extend (11) to let the travel time per unit distance \( \bar{\tau}(t) \) depend on the location or distance of the segment along the link, thus

\[
\bar{\tau}(t, z) = \bar{h}(w(t), z), \quad (11')
\]

where \( z \) is the distance from the beginning of the link to the beginning of the segment \( \Delta z \), and \( \bar{\tau}(t, z) \) is the travel time per unit distance on segment \( [z, z+\Delta z] \) for vehicles entering it at time \( t \). This allows ‘inhomogeneous’ links, whereas the whole-link model (5) implicitly assumes a homogeneous link. Similarly, we can also extend (11) to let the segment travel-time per unit distance depend on the time \( t \) at which a vehicle arrives at the beginning of the segment, thus

\[
\bar{\tau}(t, z) = \bar{h}(w(t), t, z). \quad (11'')
\]

Suppose we have computed the travel times up to some point \( z \) on the link and computed the (out)flows \( v(t) \) from the link section \([0, z]\). Then apply the model (5)–(7) to the next segment of the link, of length \( \Delta z \). At time \( t \) for segment \([z, z+\Delta z]\) let \( u_\Delta(t, z) \) and \( v_\Delta(t, z, \Delta z) \) denote the inflow and outflow rates respectively for the segment, \( \tau_\Delta(t, z) = \Delta z \bar{\tau}(t, z) \) denote the segment travel time for vehicles entering the segment at time \( t \), \( x_\Delta(t, z) \) denote the number of vehicles on the segment, and \( k_\Delta(t, z) = x_\Delta(t, z)/\Delta z \) denote the mean density on the segment.

Applying the model (5)–(7) to the segment \([z, z+\Delta z]\), using (11'') in place of (5), gives

\[
t_\Delta(t, z) = \Delta z \bar{h}(w_\Delta(t, z), t, z), \quad (5')
\]

\[
w_\Delta(t, z) = \beta u_\Delta(t, z) + (1-\beta) v_\Delta(t + \tau_\Delta(t, z), z + \Delta z), \quad (6')
\]

\[
v_\Delta(t + \tau_\Delta(t, z), z + \Delta z) = \frac{u_\Delta(t, z)}{1 + \partial \tau_\Delta(t, z)/\partial t}. \quad (7')
\]

Though (5')–(7') is the complete discretised model, two further equations implied by (7') are useful below, namely the discretised forms of (9) and (10). Rearranging (7') and integrating gives

\[
x_\Delta(0, z) + \int_0^t u_\Delta(s, z) \, ds = \int_0^{t+\tau_\Delta(t, z)} v_\Delta(s, z + \Delta z) \, ds \quad (9')
\]

and from Proposition 1

\[
x_\Delta(t, z) = x_\Delta(0, z) + \int_0^t (u_\Delta(s, z) - v_\Delta(s, z + \Delta z)) \, ds. \quad (10')
\]

We wish to compare this discretised version of the \( \tau(t) = h(w(t)) \) model with the LWR model. The basic version of the latter is based on the assumption that, on a homogeneous link, the flow rate \( q(t, z) \) at time \( t \) at each point \( z \) depends only on the density \( k(t, z) \) at that point, and not on any earlier or later points, thus \( q(t, z) = Q(k(t, z)) \). For a link that is inhomogeneous over space \( z \) and time \( t \) this becomes

\[
q(t, z) = Q(k(t, z), t, z) \quad (12)
\]

together with a conservation equation

\[
\frac{\partial q(t, z)}{\partial z} = - \frac{\partial k(t, z)}{\partial t}. \quad (13)
\]
Proposition 2. Applying the model (5)–(7) to a segment \([z, z + \Delta z]\) of a link gives (5′)–(7′) and (9′). As the segment length \(\Delta z \to 0\) these equations converge in the limit to the LWR model ((12) and (13)). That is,

(a) the travel time equations (5′) and (6′) converges to the flow-density equation (12) and
(b) the conservation equation (7′) converges to (13).

Remark. When the flow rate and speed are constant along a link (or link segment) and constant over time then: (number of vehicles on the link) = (flow rate) \times (link travel time). In the proof we first show that an analogous relationship (i.e. (14)) holds when these quantities vary as defined by the link segment model (5′)–(7′).

Proof. (a) The flow-density equation. Adding (9′) and (10′) gives \(x_\Delta(t,z) = \int_t^{t+\tau_\Delta(t,z)} v_\Delta(s,z+\Delta z) \, ds\). Using the mean value theorem, this integral implies that there exists an (out)flow \(v_\Delta(s,z\Delta z)\), at some time \(s, t \leq s \leq t + \tau_\Delta(t,z)\), such that

\[
x_\Delta(t,z) = v_\Delta(s,z+\Delta z) \tau_\Delta(t,z).
\]

Using (5′) to substitute for \(\tau_\Delta(t,z)\) in (14) gives \(x_\Delta(t,z) = v_\Delta(s,z+\Delta z) \tilde{h}(w_\Delta(t,z), t, z) \Delta z\) and using the definition of mean density \(k_\Delta(t,z) = x_\Delta(t,z)/\Delta z\) to substitute for \(x_\Delta(t,z)\) gives

\[
k_\Delta(t,z) = v_\Delta(s,z+\Delta z) \tilde{h}(w_\Delta(t,z), t, z).
\]

As \(\Delta z \to 0\), \(t + \tau_\Delta(t,z) \to t\), hence \(s \to t\) and \(v_\Delta(s,z+\Delta z) \to u_\Delta(t,z)\) which can be written as \(q(t,z)\). This also implies \(w_\Delta(t,z) \to u_\Delta(t,z) = q(t,z)\). Thus \(\Delta z \to 0\) reduces (15) to

\[
k(t,z) = q(t,z) \tilde{h}(q(t,z), t, z).
\]

Let \(R(q,t,z)\) denote \(q \tilde{h}(q,t,z)\) so that (16) can be rewritten as \(k(t,z) = R(q,(t,z), t, z)\). There exists a unique function (12) whose inverse is \(k(t,z) = R(q(t,z), t, z)\).

(b) The conservation equation. Differentiating (10′) w.r.t. \(t\) gives \(\partial x_\Delta(t,z)/\partial t = u_\Delta(t,z) - v_\Delta(t,z+\Delta z)\) and differentiating the mean density definition \(k_\Delta(t,z) = x_\Delta(t,z)/\Delta z\) gives \(\partial x_\Delta(t,z)/\partial t = \Delta z \partial k_\Delta(t,z)/\partial t\). Substituting the latter in the former yields \(\Delta z \partial k_\Delta(t,z)/\partial t = u_\Delta(t,z) - v_\Delta(t,z+\Delta z)\). Dividing both sides by \(\Delta z\) and letting \(\Delta q(t,z)\) denote the change in the flow rate from the entrance to exit of the segment, i.e., \(-\Delta q(t,z) = u_\Delta(t,z) - v_\Delta(t,z+\Delta z)\), reduces the previous equation to

\[
\frac{\partial k_\Delta(t,z)}{\partial t} = -\frac{\Delta q(t,z)}{\Delta z}.
\]

Then letting \(\Delta z \to 0\), the mean density \(k_\Delta(t,z) \to k(t,z)\) and \(\Delta q(t,z)/\Delta z \to \partial q/\partial z\), and the above equation goes to (13). □

Remark. It is of interest that, as an intermediate step, the above proof shows that the discretised link model (5′)–(7′) implies discrete equations (16) and (17), for segment \([z, z + \Delta z]\), directly analogous to the continuous equations (12) and (13). There are other ways to prove part (a) of the above proposition, without using the variable \(x(t,z)\). We can show that when \(\Delta z \to 0\) the model (5′)–(7′) implies that at \((t,z)\) speed is a continuous function of the flow rate and this continuous
function implies the continuous flow-density relationship (12). However, an advantage of the above proof is that it first shows that the model (5)–(7) implies a discrete form of the flow-density equation (12) for the segment \([z, z + \Delta z]\).

2.2.2. Corrected flow capacity for the \(\tau(t) = h(w(t))\) model (5)–(7), or (8)

Suppose that in the travel time function \(\tau = h(w)\) the maximum of \(w\) is \(w = q_c\), as illustrated in Fig. 1. \(q_c\) is called the flow capacity, or link capacity. Assuming the link is homogeneous, the flow capacity \(q_c\) should apply all along the link, including at the entrance and exit. Hence the outflow rate \(v(t + \tau(t))\) obtained from (5)–(7) should not exceed \(q_c\). However, we find that the outflow can exceed the capacity for the following reason. By definition, the argument of \(\tau = h(w)\) can not exceed \(q_c\). But, from (6), \(w\) is a weighted average of the inflow and outflow rates, hence the outflow rate may exceed \(q_c\) so long as the weighted average \(w\) does not exceed \(q_c\). In numerical solutions of (5)–(7) we found that the outflow rate would sometimes exceed \(q_c\) if there were a rapid fall in the inflow rate. Note that this issue did not arise in the original Carey et al. (2003) statement of the model, since they assumed the travel time function \(\tau = h(w)\) was defined for all \(0 \leq w \leq +\infty\) and in that case there is no capacity limit, or rather \(q_c = +\infty\). For example, if the travel time \(\tau = h(w)\) is a linear, quadratic, polynomial or exponential function over the range \([0, +\infty]\), then there is no capacity limit, and in that case ensuring \(v(t + \tau(t)) \leq q_c\) is not an issue.

The above problem of outflow, computed from (5)–(7), exceeding the capacity limit \(q_c\) inherent in \(\tau = h(w)\) arises because of the approximation involved in using finite length segments. If we used infinitesimally small segments then the outflow from the segment would be arbitrarily close to the inflow to the segment, and since the inflow does not exceed the capacity \(q_c\), the outflow would not exceed the capacity, hence the problem referred to above would not arise. Another way to see this is to note that for infinitesimal segments the Carey et al. model is equivalent to the LWR model (as shown in Section 2.2.1), and of course the latter ensures that flow does not exceed capacity.

![Fig. 1. Scenarios for \(\tau = h(w)\).](image-url)
In what follows, we adapt or correct the model (5)–(7) to ensure that the computed outflows $v(t + \tau(t))$ respect the flow capacity, that is, satisfy $v(t + \tau(t)) \leq q_c$. Basically, the adaptation consists of ensuring that if the outflows at some time hit the capacity or ceiling $q_c$ then they continue at the capacity level rather than rising above it. The scheme seems consistent with the discussion in say Newell (1988). It keeps traffic moving without exceeding the allowable maximum flow.

We note in passing that there are then three possible scenarios for a flow capacity for $\tau = h(w)$, as illustrated in Fig. 1: (a) $\tau = h(w)$ is asymptotic to the bound $q_c$, (b) $\tau = h(w)$ achieves the bound and then bends backwards, or (c) $\tau = h(w)$ achieves the bound and then becomes vertical line.

To ensure that $v(t + \tau(t)) \leq q_c$ will always hold, we can adjust the model (5)–(7), as follows. Redefine $v(t + \tau(t))$ as $v(t + \tau(t)) = \min\{q_c, v(t + \tau(t))\}$ given by (5)–(7). This can only reduce the outflow rate $v(t + \tau(t))$ hence delay exit times $t + \tau(t)$. The new travel times $\tau(t)$ can be obtained from (5)–(7) as follows. Suppose that at some time, say $t^*$, the model (5)–(7) yields $v(t + \tau(t)) > q_c$. Then set

$$v(t + \tau(t)) = q_c$$

hence, on rearranging (7), we have

$$\tau'(t) = u(t)/q_c - 1.$$  \hspace{1cm} (19)

Integrating $\tau'\cdot t$ from time $t = t^*$ gives

$$\tau(t) = \tau(t^*) + \int_{t^*}^{t} (u(s)/q_c - 1) \, ds, \quad t \geq t^*.$$  \hspace{1cm} (20)

Thus if at some time $t^*$ the model (5)–(7) yields $v(t + \tau(t)) > q_c$, then temporarily switch from the model (5)–(7) to (18)–(20). Switch back to the (5)–(7) as soon as that model would again yield a $v(t + \tau(t)) \leq q_c$. To determine when to switch back, keep track of the value of $v(t + \tau(t))$ that would be obtained if at time $t$ we switched back to (5)–(7). Call this value $\tilde{v}(t + \tau(t))$. To obtain $\tilde{v}(t + \tau(t))$, invert (5) and (6), thus

$$\tilde{v}(t + \tau(t)) = \left( h^{-1}(\tau(t)) - \beta u(t) \right) / (1 - \beta),$$  \hspace{1cm} (21)

where $\tau(t)$ in (21) is the current value of $\tau(t)$ from (20), and $u(t)$ is given. Switch back to (5)–(7) if $\tilde{v}(t + \tau(t)) \leq q_c$. This switching process is much easier than it seems, since in practice the model is not solved as continuous functions but by discretising time into very small time steps. At each time step we can easily check whether $v(t + \tau(t)) > q_c$, if we are currently using (5)–(7), and check whether $\tilde{v}(t + \tau(t)) \leq q_c$, if we are using (18)–(20).

The backward bending $\tau = h(w)$ corresponds to a downward sloping flow-density function. In the LWR model the downward sloping part of the flow-density function is never attained if the link is homogeneous and there are no obstacles or stop signals to restrict flow. It is therefore desirable, in this scenario, that in any solution of (5)–(7) the backward bending part of the $\tau = h(w)$ would not be attained. The following Proposition shows that this is true if (5)–(7) is corrected as set out above.

**Proposition 3.** Suppose the travel time function $\tau(t) = h(w(t))$, as in model (5)–(7) or (8), is backward bending, as illustrated in Fig. 1. Let $(q_c, \tau_c)$ denote the turning point at which $\tau = h(w)$ bends backwards, and assume that
(i) Inflows $u(t)$ satisfy $u(t) \leq B < q_c$.
(ii) $v(t + \tau(t)) \leq q_c$ for all $t$, which can be ensured by switching between (5)–(7) and (18)–(20) as outlined above.

Then in any solution of (5)–(7) or (8), the travel times $\tau(t)$ will always be on the lower branch of the backward bending curve $\tau = h(w)$ for all $t$ in $[0, T]$.

**Remark.** Recall that model (5)–(7) or (8) implicitly assumes that the link is homogeneous, and we have assumed there are no obstructions or flow controls.

The assumption that $\tau = h(w)$ is ‘backward bending’ can be stated formally here as: let $\tau = h(w)$ be nonnegative and continuous with $h(0) > 0$; let $\tau = h(w)$ be increasing (or nondecreasing) from $w = 0$ up to some $w = q_c$, at which point $h(w) \text{‘bends backwards’}$; and let the upper, backward bending, branch of $\tau = h(w)$ be decreasing (or nonincreasing) in $w$.

**Proof.** In any solution of (5)–(7), or (8), $\tau(t)$ could move onto the backward bending part of $\tau = h(w)$ only by either,

(a) moving continuously along the line $\tau = h(w)$ from the upward sloping part to the backward bending part, which requires $w(t)$ attaining the value $w(t) = q_c$, or by

(b) jumping discontinuously from the lower to the upper branch.

Case (a) can be ruled out as follows. By assumption (ii), $v(t + \tau(t)) \leq q_c$. Also, by assumption, $u(t) < q_c$ hence $w(t) = [\beta u(t) + (1 - \beta)v(t + \tau(t))] < q_c$, since $0 \leq \beta < 1$. That is, the bound $w(t) = q_c$ cannot be attained, so that $w(t)$ varies only in the range $0 \leq w(t) < q_c$.

Case (b) can be ruled out since, if for any given value of $w(t)$ there are two values of $\tau(t)$, we can always simply choose the value of $\tau(t)$ on the lower branch of $\tau = h(w)$.

3. Models used in the numerical examples

We wish to compare the solutions from the above travel time models and compare these with LWR model, which is based on a flow-density equation. To make meaningful comparisons between these three models note that all three should yield the same results when flows and travel times are constant over time. In that case the three models are $\tau = f(x)$, $\tau = h(w)$ and $q = Q(k)$, and the number of vehicles on a link is $x = w\tau$ and is also given by $x = Lk$. We use these relationships to make transformations between the three models when flows and densities are constant. Specifically, we obtain $\tau = h(w)$ and $q = Q(k)$ models from given $\tau = f(x)$ models; of course, the inverse process is also feasible. Later in this section we show how to solve the $\tau = h(w)$ model.

First, let us set out particular forms for the travel time models $\tau = f(x)$ and $\tau = h(w)$ set out in Sections 2.1 and 2.2, respectively. We chose linear and quadratic forms of $\tau = f(x)$ and derive the corresponding forms of $\tau = h(w)$ as set out below. Also, since the choice of parameter values for the travel time models affects the results, their values should represent physically realistic or
typical behaviour. The values given below are also used in Carey and Ge (2001) where they are shown to be realistic. These forms and parameter values are used in the numerical examples in Section 4.

3.1. Using a linear form of $\tau = f(x)$

The earliest form of $\tau = f(x)$ used in DTA (Friesz et al., 1993) was the linear form, which can be stated as

$$\tau = a + bx. \quad (22)$$

We assume $a = 1.1$ min, $b = 0.02$ min/veh, and also assume a link length $L = 1.2$ km. For simplicity of exposition, the units will be omitted in the later presentation.

3.1.1. The flow-density function $q = Q(k)$

To obtain the corresponding flow-density function, for the LWR model, substitute $x = Lk$ and $\tau = x/q = Lk/q$ in (22), and rearrange, thus

$$q = Lk/(a + bLk) \quad (23)$$

which is a concave function that passes through the origin and has an upper asymptote, $q = 1/b$.

3.1.2. The weighted inflow–outflow model $\tau(t) = h(w(t))$

Using $x = w\tau$ to substitute for $x$ in (22) and rearranging gives $\tau = a/(1 - bw)$ hence

$$\tau(t) = a/(1 - bw(t)) \quad (24)$$

which is increasing in $w$, starting from $(w, \tau) = (0, a)$ and with a vertical asymptote at $w = 1/b$. Recall from Section 2 that in the travel time model $\tau = h(w)$, the ‘flow’ argument $w$ is defined by (6), i.e., $w(t) = \beta u(t) + (1 - \beta)v(t + \tau(t))$, hence using (7) gives $w(t) = u(t)(1 + \beta\tau'(t))/(1 + \tau'(t))$. Substituting this expression for $w(t)$ into (24) and rearranging gives

$$g(\tau'(t), \tau(t), u(t)) = 0, \quad (25)$$

where

$$g(\tau'(t), \tau(t), u(t)) = bu(t)\tau(t)((1 + \beta\tau'(t))/(1 + \tau'(t))) - \tau(t) + a.$$ 

We can solve this differential equation for $\tau(t)$ starting from some initial solution $\tau(0)$, as discussed in Section 3.3.

3.2. Using a quadratic form of $\tau = f(x)$

A quadratic travel time function is used in Wu et al. (1995, 1998) and in Xu et al. (1999) and can be written as

$$\tau = a + bx + cx^2. \quad (26)$$

As in the linear case, we assume $a = 1.1$ min, $b = 0.02$ min/veh and a link length $L = 1.2$ km. We also assume $c = 10^{-4}$ min/veh², as in Carey and Ge (2001).
3.2.1. The flow-density function $q = Q(k)$

Substituting $x = Lk$ for $x$ and $\tau = x/q = Lk/q$ for $\tau$ in (26) and rearranging gives

$$q = Lk/(a + bLk + cL^2K^2).$$

(27)

This concave function starts at $(k, q) = (0, 0)$, increases to a peak at

$$(k_c, q_c) = ((1/L(\sqrt{ac}/c, 1/(2\sqrt{ac} + b))) \approx (55.2, 24.4)$$

(28)

and then bends downwards (with negative gradient) to approach the horizontal $k$ axis asymptotically. The function thus has two values of $k$ for each value of $q(0 < q < q_c)$, but a single value of $q$ for each $k$.

3.2.2. The weighted inflow–outflow model $\tau(t) = h(w(t))$

When the flow, density and travel time are constant over time $x = q\tau$ and substituting this in (26) and rearranging gives

$$\tau = a + bq\tau + cq^2\tau^2$$

(29)

or

$$cq^2\tau^2 - (1 - bq)\tau + a = 0.$$

(30)

Solving this equation for $\tau$ yields the travel time-flow relationship

$$\tau = \frac{(1 - bq) \pm \sqrt{(1 - bq)^2 - 4acq^2}}{2cq^2}.$$  

(31)

This starts at $\tau = a = 1.1$ when $q = 0$, increases to $(q_c, \tau_c) = (1/(2\sqrt{ac} + b), 2a + b\sqrt{ac/c}) \approx (24.4, 4.3)$ and then bends backwards (with negative gradient) to approach the vertical $\tau$ axis asymptotically. This mapping has two values of $\tau$ for each value of $q(0 < q < q_c)$.

In the travel time model $\tau = h(w)$, the ‘flow’ argument is given by $w$ defined by (6). From (6) and (7), $w(t) = u(t)(1 + \beta\tau(t))/(1 + \tau(t))$ and substituting this for $q$ in (31) (or (29) and (30)) gives

$$g(\tau'(t), \tau(t), u(t)) = 0,$$

(32)

where

$$g(\tau'(t), \tau(t), u(t)) = c\tau^2(t)\left[\beta u(t)\frac{\tau'(t) + 1/\beta}{\tau'(t) + 1}\right]^2 + b\tau(t)\left[\beta u(t)\frac{\tau'(t) + 1/\beta}{\tau'(t) + 1}\right] - \tau(t) + a.$$  

Again we can solve this differential equation for $\tau(t)$ as discussed in Section 3.3.

Now it is necessary to discuss how to calculate travel time using (29) for a discretised link. Consider a homogeneous link of length $L$, which is divided into $n$ segments of length $L_i (i = 1, 2, \ldots, n)$. To apply the travel time equation to each segment we simply adjust the travel time equation (29) so that it applies to segment rather than a whole link and then proceed as before. Eq. (29) relates the flow rate $q$ to the travel time $\tau$ on the whole link. Since equation (29) was derived assuming uniform flow on the link, the travel time for segment $i$ is a fraction $\theta_i = L_i/L$ of this, hence $\tau_i = \theta_i\tau$ and $\tau = \tau_i/\theta_i$. Substituting this in (29) gives

$$\tau_i/\theta_i = a + bq_i\theta_i/\theta_i + c(q_i\theta_i/\theta_i)^2,$$

(29')
where \( q_i \) is the flow rate on the segment. We denote the segment flow rate \( q_i \) so we can later let flow rates vary over segments. It is convenient to rewrite (29') to include the parameter \( \theta_i = L_i/L \) in the other parameters, thus

\[
\tau_i = a^\lambda + b^\lambda q_i \tau_i + c^\lambda(q_i \tau_i)^2, \tag{29''}
\]

where \( a^\lambda = a \theta_i, b^\lambda = b \) and \( c^\lambda = c/\theta_i \). With this redefinition of the parameters, equation (29'') is of exactly the same form as (29). The steps needed to calculate travel times using (29) for a whole link have already been set out above. This can now be applied to each link segment, changing only the parameter definitions for \( a, b \) and \( c \).

A similar discussion applies to the linear form (24). Thus, substituting \( \tau = \tau_i/\theta_i \) in \( \tau = a/(1 - b q) \) and replacing \( q \) with \( q_i \) gives \( \tau_i = a \theta_i/(1 - b q_i) \), which is of the same form as (24) with the parameter \( a \) redefined as \( a \theta_i \). Thus the differential equation (25) can be applied to each link segment with \( a \theta_i \) and \( q_i \) replacing \( a \) and \( q \).

### 3.3. Computing travel time using \( \tau(t) = h(w(t)) \) and given inflows \( u(t) \)

From the definitional expression (5)–(7) of model \( \tau(t) = h(w(t)) \) we can obtain a first-order differential equation \( g(\tau'(t), \tau(t), u(t)) = 0 \), as obtained in (25) and (32). This subsection discusses how to solve equations of this class. Generally it is difficult to find the analytic solution of this kind of problem therefore a numerical solution method is used. To this end, substituting \( [\tau(t) - \tau(t - \Delta t)]/\Delta t \) for \( \tau'(t) \) in \( g(\tau'(t), \tau(t), u(t)) \)

\[
g(\tau(t) - \tau(t - \Delta t))/\Delta t, \tau(t), u(t)) = 0. \tag{32}
\]

At time \( t \), given \( \Delta t, u(t) \) and \( \tau(t - \Delta t) \), the only unknown in the above equation is \( \tau(t) \) so we can rewrite (32) as \( g(\tau(t)) = 0 \). Usually this equation is nonlinear and it is not easy to find an analytical solution. In mathematics there are many methods available for solving this problem (Polak, 1997). We used the local Newton’s method to solve numerically for \( \tau(t) \), as follows. Given a current approximation \( \tau^m(t) \) to a solution of (32) at iteration \( m \), the local Newton’s method gives the next, and potentially better, approximation to a solution of (32) as follows

\[
\tau^{m+1}(t) = \tau^m(t) - g(\tau^m(t))/g'(\tau^m(t)). \tag{33}
\]

As \( m \to \infty \), \( \tau^m(t) \) approaches a solution for \( \tau(t) \). As a convergence criterion we used the relative change rate from two successive iterations or approximations, i.e., \( |\tau^{m+1} - \tau^m|/\tau^m \leq \epsilon \), where \( \epsilon \) is a tolerance. This is the simplest form of the Newton’s method for solving the root finding problem. It is always successful if initialised sufficiently close to a solution of (32). We assumed an initial solution \( \tau^0(t) = \tau(t - \Delta t) \).

### 4. Numerical comparisons of the two models

In this section we apply the travel time models from the previous section to inflow profiles set out below. We describe time as continuous, but since the solution method is based on discrete approximations, instead of continuous time we in fact use a very fine discretisation of time, over the time span \([0, T]\). We divide space (the link length) into \( n \) segments, and apply the \( \tau = f(x) \)
model to each link segment in succession as described in Section 2. Similarly for the $\tau = h(w)$ model. The results are shown in Figs. 2–4 and Tables 1–4 below. In these figures the inflow $u(t)$ is the inflows to the first segment of the link at time $t$ and the outflow $v(t)$ is the outflows from the final ($n^{th}$) segment of the link at time $t$. The link is empty at the initial time $t = 0$. The travel time $\tau(t)$ is the time taken to travel from entering the first segment to exiting from the final segment.

In Section 2.2.1 we allow the link to be inhomogeneous, by allowing the characteristics of the link segments to differ. However, in the examples below we assume a homogeneous link for the following reason. The original travel time models, set out in Sections 2.1 and 2.2, implicitly assume homogeneity along the link, and that is also true of other ‘whole link’ models. In this paper we wish to examine how the link traversal times and outflows are affected by dividing the link into segments. If at the same time we introduced inhomogeneity along the link, then the results from

Fig. 2. How discretisation affects solution profiles in Example 1: (a) outflow profiles without bounds, (b) travel time profiles without bounds on outflows, (c) outflow profiles with bounds and (d) travel time profiles with bounds on outflows.
the segmented link may not be comparable with the results for the whole undivided link. Also, the existing literature on the above travel time models does not discuss inhomogeneity over time or space. Further, even when dividing a link into segments, it is usual to consider the homogeneous case first. For example Daganzo (1995) examines the homogeneous case and then notes that the results can be extended to an inhomogeneous highway (e.g., with varying width and time-dependent conditions). He also notes that it should be sufficient (and easier) to note that any space-inhomogeneities arising in real life are approximated by a piecewise homogeneous highway, and that the results applied to the homogeneous pieces of such a highway.

In all numerical example we assume $\beta = 0.5$.

### 4.1. Comparisons using a linear $\tau = a + bx$ function

We first used the linear travel time function $\tau = a + bx$ set out in Section 3.1 and the corresponding $\tau = h(w)$ model.
Table 1
Mean percentage deviation of the **outflow** profiles from the LWR solution in Fig. 2(a) and (c)

<table>
<thead>
<tr>
<th>Number of segments $n$</th>
<th>1</th>
<th>2</th>
<th>10</th>
<th>550</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau = f(x)$</td>
<td>8.4836</td>
<td>5.7769</td>
<td>1.7224</td>
<td>0.1559</td>
</tr>
<tr>
<td>$\tau = h(w)$ without bound on $v(t)$</td>
<td>2.1837</td>
<td>1.2647</td>
<td>0.3308</td>
<td>0.1983</td>
</tr>
<tr>
<td>$\tau = h(w)$ with bound on $v(t)$</td>
<td>1.6485</td>
<td>0.8552</td>
<td>0.2566</td>
<td>0.1983</td>
</tr>
</tbody>
</table>

Table 2
Mean percentage deviation of the **travel time** profiles from the LWR solution in Fig. 2(b) and (d)

<table>
<thead>
<tr>
<th>Number of segments $n$</th>
<th>1</th>
<th>2</th>
<th>10</th>
<th>550</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau = f(x)$</td>
<td>10.4243</td>
<td>5.9722</td>
<td>1.2992</td>
<td>0.0134</td>
</tr>
<tr>
<td>$\tau = h(w)$ without bound on $v(t)$</td>
<td>1.4010</td>
<td>0.5237</td>
<td>0.0535</td>
<td>0.0304</td>
</tr>
<tr>
<td>$\tau = h(w)$ with bound on $v(t)$</td>
<td>1.1357</td>
<td>0.3523</td>
<td>0.0377</td>
<td>0.0304</td>
</tr>
</tbody>
</table>

Fig. 4. How discretisation affects solution profiles in Example 3: (a) outflow profiles and (b) travel time profiles.
Example 1. With plateau shaped inflow profile.

In this example we assumed inflows rise over time from zero to a peak at 32, remain there for a time and fall off again asymptotically to 20. The inflow profile is chosen, as described in Carey and Ge (2001), to avoid sharp changes in the inflow rate. More specifically, we let

$$ u(t) = \begin{cases} 
32 \sin(\pi t/10), & 0 \leq t < 5, \\
32, & 5 \leq t < 10, \\
20 + 12 \sin^2(\pi(t+4)/28), & 10 \leq t \leq 24 
\end{cases} $$

in veh/min. We applied the models from Section 3.1 to these inflows and computed the link travel times and outflows at each point in time, using a fine discretisation of time (dividing the time horizon $[0, 24]$ into 12,000 time intervals). We then divided the link into $n = 2$ identical segments and applied the model again, sequentially to each segment, as described in Section 2 above. We repeated this using $n = 2, 10, 50, 100, \ldots, 550$ segments, and in each case computed the profiles of outflows and travel times.

As the number of link segments $n$ increases, the link outflow profiles converge to a limit profile as shown in Fig. 2(a) and the travel time profiles converge to a limit profile as shown in Fig. 2(b). To avoid clutter, only a few graph lines are displayed, corresponding to $n = 1, 2, 10, 550$. It can be seen that the limit profiles, for both outflow and travel time, are identical to the profiles obtained from the LWR model, and indeed the solution profiles converge to very close to their limit profiles long before the maximum number of segments (550) is reached. The LWR solution illustrates the typical characteristics of that model. When the inflow $u(t)$ is increasing with time $t$, the traffic outflow rate $v(t)$ increases more gradually. When the inflow rate is decreasing a discontinuous fall
in outflow, a shock wave, is observed (Newell, 1988). In Fig. 2(a) we can see such a discontinuity in outflows at about time \( t = 19 \).

In all Figs. 2–4 the graph lines corresponding to \( n = 550 \) link segments are not visible since they coincide with the LWR graph lines.

The solution profiles (for outflows and travel times) obtained using the two travel time models can be compared visually in Fig. 2(a) and (b). However, it is difficult to distinguish some of the lines in the figures since they are so close together. Partly because of that, we have computed measures of convergence to the limit profile (the LWR profile), in Tables 1 and 2. These tables give the mean percentage deviation (MPD) of the outflow profile from the limit profile. Let \( v^n(t) \) denote the outflow computed using \( n \) link segments and \( v^l(t) \) denote the outflow computed in the limit using a large number of link segments (or computed using the LWR model). Then

\[
\text{MPD} = \frac{100}{T} \sum_{t=1}^{T} \left( \frac{|v^n(t) - v^l(t)|}{v^l(t)} \right)
\]

where \( T \) denotes the number of time intervals. In the summation we do not include \( v^n(t) \) for the initial zero outflows, to avoid a zero denominator. Similarly we computed the MPD for travel times, thus

\[
\text{MPD} = \frac{100}{T} \sum_{t=1}^{T} \left( \frac{|s^n(t) - s^l(t)|}{s^l(t)} \right)
\]

The results in Tables 1 and 2 again show that for both models only a relatively small number of segments are needed to obtain a solution that is close to the limit solution profile.

It can be seen from the tables that for any given number of segments \( n \), the solution profile for the \( \tau = h(w) \) model is much closer to the LWR limit than is the solution profile for the \( \tau = f(x) \) model. This is the most obvious for the profiles of link travel times. For example, in Table 1, when using 1, 2 or 10 segments, the outflow profiles for the \( \tau = h(w) \) model deviate from the LWR by only 1/4 as much as the \( \tau = f(x) \) model. Similarly, in Table 2, the travel time profiles for the \( \tau = h(w) \) model deviate by between 1/7 and 1/34 as much as the \( \tau = f(x) \) model. When a very large number of segments (e.g. 550) are used, all solutions are very close to the LWR solution and there is very little difference between them: the outflows deviate by less than 0.2% and the travel times by less than 0.04%.

4.1.1. Correcting the outflow rate \( v(t) \) as in Section 2.2.2

In the above experiments we used the \( \tau = h(w) \) model as set out in Section 2.2. But as discussed in Section 2.2.2, this allows the outflow rate \( v(t) \) to exceed the maximum inflow rate and possibly the capacity flow rate. In Fig. 2(a) we see fluctuations or “spikes” in the outflows for the \( \tau = h(w) \) model, just before the sharp decline in the outflow rate. These fluctuations do not appear to reflect real traffic behaviour, but instead are caused by the \( \tau = h(w) \) model adjusting too quickly and overreacting or overcompensating. As explained in Section 2.2.2, this can occur if there is a rapid fall in inflow rate. To reduce or prevent this phenomenon, we reran these experiments with the outflow correction procedure described in Section 2.2.2. The results are shown in Fig. 2(c) and (d). We see that introducing the bound on \( v(t) \) has eliminated the undesirable fluctuations in \( v(t) \), and from Table 1(a) and (b) we see that it has also moved the solution profile closer to the limit profile. The above fluctuations or spikes can also be reduced by using a smaller value of \( \beta \): we usually used a \( \beta = 0.5 \), but if we reduced this towards zero the unwanted fluctuations disappeared. Also, for any given segment lengths, a value of \( \beta \) between 0 and 0.5 gave a solution closer to the limit solution (the LWR solution).

We found similar results when we experimented with other inflow profiles or used different parameters for the travel time functions.
Example 2. With hill shaped inflow profile.

In the example above, the extended flat peak allowed the outflow and travel time profiles to converge to the flat peak. In the present example we remove the flat peak, by simply letting the inflows take longer to build up to the peak. More specifically, we let

$$u(t) = \begin{cases} 
32 \sin(\pi t / 20), & 0 \leq t < 10, \\
20 + 12 \sin^2(\pi (t + 4) / 28), & 10 \leq t < 24.
\end{cases}$$

(35)

We repeated the same experiments as in Example 1 above, using the same travel time models from Section 3.1. We again used the outflow correction procedure from Section 2.2.2 for $v(t)$ and again found that this yielded solutions closer to the limit profile. The resulting outflows and travel times are shown in Fig. 3(a) and (b). These again illustrate that, as the number of link segments $n$ increases, the solutions converge to a solution that is also the solution of the LWR model. Again the solution profiles (for outflows and travel times) for the $\tau = h(w)$ model are much closer to the LWR limit than those for the $\tau = f(x)$ model, as can be seen from Table 3.

4.2. Comparisons using a quadratic $\tau = f(x)$ function

In the above examples we assumed linear travel time functions. To show that similar convergence results are obtained using a nonlinear travel time function, we here assume a quadratic function $\tau = a + bx + cx^2$, and the corresponding $\tau = h(w)$ model from Section 3.2.

Example 3. With plateau shaped inflow profile

We assume that the inflow profile for the link is

$$u(t) = \begin{cases} 
22 \sin(\pi t / 10), & 0 \leq t < 5, \\
22, & 5 \leq t < 10, \\
10 + 12 \sin^2(\pi (t + 4) / 28), & 10 \leq t < 24.
\end{cases}$$

(36)

in veh/min. We repeated the experiments from Examples 1 and 2, using the above inflows and travel time functions. We again used the outflow correction procedure from Section 2.2.2 for $v(t)$ and again found that this yielded solutions closer to the limit profile. The solutions, shown in Fig. 4(a) and (b), again illustrate that, as the number of link segments $n$ increases, the solutions from both models converge to a solution that is also the solution of the LWR model. Also, the solution profiles for the $\tau = h(w)$ model are again much closer to the LWR limit than those for the $\tau = f(x)$ model, as shown in Table 4.

We also experimented with several other inflow profiles and found similar results.

5. Concluding remarks

This paper considers two link travel time models that have been used to describe link travel times in the DTA models. Though the models have generally been applied to whole links, we here
divide the link into segments and apply the models sequentially to the segments. For given inflow profiles we compute solution profiles for travel times and outflows. As the segment lengths are reduced, the solution profiles from the two models converge to the same solution profile, which is the LWR solution. We applied the two models to various numerical examples based on realistic parameter values for travel time functions and inflows. On comparing the solution profiles from the two models we found that, when using the same number of link segments for each, the second model (the \( \tau(t) = h(w(t)) \) model) gave solutions significantly closer to the limit solution. Even when the link is not discretised the solution profiles for the \( \tau(t) = h(w(t)) \) model are within a few percent of the limit solution.

Though in the numerical examples the \( \tau(t) = h(w(t)) \) model yields solutions that (for the same number of link segments) are much closer to the limit profile, this does not prove that this always holds. Also, the \( \tau(t) = f(x(t)) \) model gives solutions equally close to the LWR solution by simply refining the discretisation. Further, using a smaller number of segments to achieve the same accuracy may not be the only factor in deciding which of the two models to use in DTA. For example, the way in which the model can be integrated into the network model for DTA may be another factor in deciding which model to use. Our purpose here is only to explore and compare how the models behave.

In Section 2.2.1 we show that, as the segment lengths are reduced, the models converge even when the link segments are not homogeneous, that is, have different flow capacities. In Section 2.2.2 we introduced a procedure to correct for an approximation error in the \( \tau(t) = h(w(t)) \) model that is due to using finite segment lengths: the error would allow the computed outflow to exceed the capacity inherent in the travel time function. Correcting this error improves the approximation of the model to the LWR model. This is confirmed in the numerical examples in Section 4.1, where the correction scheme reduces unwanted fluctuations in outflow that the second model can otherwise produce in certain cases (in particular, if there is a rapid fall in inflows).

In the discussion and numerical examples in this paper we assume that time is treated as continuous, since it is normally treated as continuous in the DTA literature where these whole-link travel time models are used. Actually, both here and in the literature, time is divided into small steps to enable computation of numerical solutions, but the time steps are very small. We can easily discretise time, as well as the link length, into larger steps, and we consider that in a separate note. If the link segments and the time steps are chosen in a particular way, the solutions obtained are closer to the LWR solution. That is, choose time steps of length equal to the free flow travel time for the link segment. If the time steps and segment lengths are not coordinated in this way then it usually takes a larger number of time steps or space segments to achieve the same closeness to the limit solution (the LWR solution). This result is counter to the usual intuitive assumption that treating time as continuous is more accurate.

Finally, it is worth remarking on computing costs, though this paper is not about computing costs or algorithms. The computing costs for each of the two travel-time models, and for the finite difference approximation to the LWR model, are of the same order. To see this, note that in each of the three models we perform a computation at each segment or cell of the discretised link, and repeat these computations for each time step. For simplicity and comparison, suppose we use the same discretisation in the three models. If we have \( n_L \) link segments and \( n_T \) time steps then the computational load is proportional to \( n_L n_T \), say \( C = Kn_L n_T \) where the constant \( K \) is different for each of the three models. Since the computation cost \( K \) incurred at each time-space step is
independent of the problem size $n_L$ and $n_T$, it is not as important as $n_L$ or $n_T$ when considering computing costs.

The above comparison of computing costs, for a discretised link, does not explain why link travel time models, and not a finite difference approximation to the LWR model, have been used in optimisation or variational inequality formulations of DTA for networks. However, the computing time was not the main reason why the $\tau(t) = f(x(t))$ model was introduced in DTA. It was introduced because it was shown that it could tractably be included in plausible network traffic assignment models formulated as optimisation problems or as variational inequality problems. Various authors have shown that these particular network optimisation or variational inequality formulations could be readily analysed and solved. Also, when the $\tau(t) = f(x(t))$ travel time model has been used in network assignment, the original links have typically not been discretised. Though the LWR model, and its finite difference approximation, has certain theoretical advantages there are still difficulties in tractably including it in optimisation or variational inequality formulations of traffic network assignment. An important start on that has been made by Ziliaskopoulos (2000), Lo (1999) and Lo and Szeto (2002). However, there are still problems to be solved, in particular concerned with handling multiple destinations or traffic types. Lo (1999) uses mixed-integer programming to handle that, which is computationally costly and presumably as yet practical only for relatively small problems.

Acknowledgements

The authors would like to thank two anonymous referees for their helpful comments and suggestion. This research was supported by a UK Engineering and Physical Science Research Council (EPSRC) grant number GR/R/70101, which is gratefully acknowledged.

References


Carey, M., 1999. A framework for user equilibrium dynamic traffic assignment, Research Report. Faculty of Business and Management, University of Ulster, BT37 0QB. Being revised for publication.


Carey, M., Ge, Y.E., 2001. Convergence of a whole-link travel time model. Faculty of Business and Management, University of Ulster, Northern Ireland, BT7 0QB.

Carey, M., Ge, Y.E., 2002. Alternative conditions for well-behaved travel-time functions. Faculty of Business and Management, University of Ulster, Northern Ireland, BT7 0QB.