Pseudo-periodicity in a travel-time model used in dynamic traffic assignment

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Abstract

In the past several years, in network models for dynamic traffic assignment, link travel times have frequently been treated as a function of the number of vehicles on the link. In an earlier paper, the present authors considered the linear form of this link travel-time function and showed that if there is a step increase in the inflow pattern this causes an infinite sequence of steps or jumps in the outflow profile, gradually damping out over time. This paper extends the analysis of this phenomenon to nonlinear travel-time functions and to more general inflow patterns. We show that the phenomenon occurs with general travel-time functions, and occurs whether the flow changes in discontinuous steps or more smoothly, and whether flows increase or decrease. We illustrate the results with numerical examples. We find, and prove, some surprising results, in particular that, in the travel-time model, outflows can take a much longer time to adjust to small falls in inflows than to large falls in inflows.

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1. Introduction

When modelling time-varying flows on congested traffic networks, it has often been found convenient to treat the travel time on each link as a function of the number of vehicles on the link. That is, for traffic entering a link at time \( t \), let the time taken to traverse the link be

\[ \tau(t) = f(x(t)) \] (1a)
where \( x(t) \) is the number of vehicles on the link at that time \( t \). Following Heydecker and Addison (1998) we refer to models such as (1a) as whole-link models, since the variables \( \tau(t) \) and \( x(t) \) refer to the whole link, in contrast to a model that would explicitly consider traffic flow, density and speed varying along the link. A linear form of (1a) was introduced in network models by Friesz et al. (1993), examined further in Xu et al. (1996, 1999) and has been used in other articles. That is, \[
\tau(t) = \alpha + \beta x(t) \tag{1b}
\]
where \( \alpha \) and \( \beta \) are constants. The nonlinear form (1a), as well as the linear form, was used in Fernandez and de Cea (1994), Ran et al. (1993), Ran and Boyce (1996), Astarita (1995, 1996), Wu et al. (1995, 1998), Xu et al. (1996, 1999), Adamo et al. (1999) and elsewhere.

Carey and McCartney (2002) considered the linear model (1b) and showed that it exhibited a certain form of ‘pseudo-periodicity’, that does not reflect any real behaviour of road traffic. More specifically, they showed that if there is a finite jump discontinuity in the inflow rate this is ‘copied’ from the inflow to the outflow profile, and is repeated an infinite number of times in the outflow profile, gradually damping out over time. This knock-on effect on the outflow rate also affects the time profile of link travel times.

It has been suggested that the above pseudo-periodicity or knock-on effects may be due to linearity of the travel-time functions, hence in the present paper we extend the analysis to include nonlinear travel-time functions (1a). We find that the phenomenon still occurs, though in more complex ways. This is of interest in itself, but is also of interest since there are already other types of variability in dynamic traffic assignment models, and it is increasingly important to distinguish between them. For example, (1a) may be used in DTA models that involve within-day dynamics due to time-varying inflows or control parameters, or day-to-day dynamics due to driver learning behaviour or day-to-day variation in inflows. It may include dynamics due to converging, perhaps cyclically, to an ‘equilibrium’ or due to the equilibrium itself moving over time, or simply due to the random variables in a simulation run. If we are to understand or interpret the results from such models it is important to be aware of the source of pseudo-periodicity or variability considered in this paper.

We do not in this paper consider the general case for or against using (1a) in dynamic traffic assignment models, nor compare it with possible alternative models. That is already considered in the existing literature. However, Friesz et al. and other authors using models (1a) or (1b) do recognise that they are approximations. Adamo et al. (1999), using (1a) in a network model, remarks (p. 556) that “Due to its approximate nature, this model lends itself more to general planning applications than to microapplications such as adaptive signal control.”

Others have raised issues concerned with the whole-link travel-time function \( \tau(t) = f(x(t)) \), but they have not noted or discussed the pseudo-periodic effect considered in the present paper. Daganzo (1995) argued that the travel-time function \( \tau(t) = f(x(t)) \) should not be used (or should be restricted to a constant or a straight line through the origin) since it cannot take account of the distribution of vehicles along a link. That applies almost by definition to all so-called whole-link models, and is recognised by the users of those models: they use whole-link models as an approximation (as noted above by Adamo et al.), so as to construct tractable network models for dynamic traffic assignment.

The issues in this paper are also different from those in Carey and Ge (2001a,b) though they consider the same model (1) as above. They divide the link into segments and examine how the
solution is affected by the choice of segment lengths, and choice of time steps. They note that if the discretisation of space and time is refined to the continuous limit then the solution is equivalent to that of the hydrodynamic model of Lighthill and Whitam (1955) and Richards (1956). The LWR model does not exhibit the pseudo-periodicity shown in this paper. For this and other reasons it seems that the pseudo-periodicity considered in this paper is reduced if the link lengths and the time intervals are reduced.

1.1. Linear and nonlinear forms of (1a)

Before proceeding it is worth asking what difference does a nonlinear form (1a) makes as compared to a linear form (1b). To illustrate, we here compare the properties and implications of a linear form and a quadratic form, while for simplicity assuming flows and travel times constant over time. The results for the quadratic case are also useful later in this paper. Flows varying over time are considered in the rest of the paper. Denote the constant inflow = outflow by $u$, constant number of vehicles on the link by $x$, and constant travel time by $s$. When flows are constant over time we have $x = us$ and also $x = ak$ where $k$ is traffic density and $a$ is the link length. Combining $x = us$ and $x = ak$ gives $s = ak/u$.

Linear form (1b). Substituting $x = ut$ in (1b) gives $\tau = \frac{x}{1 - \beta u}$. This is convex upward sloping from $(u, \tau) = (0, x)$, with a vertical asymptote at $u = 1/\beta$, which indicates a link flow capacity of $1/\beta$: this was not immediately obvious from (1b) but is noted in the literature (e.g. in Friesz et al. (1993)). Substituting $\tau = ak/u$ and $x = ak$ in (1b) and rearranging gives the flow–density function $u = ak/(x + a\beta k)$. This is a concave function passing through the origin, and approaching a horizontal asymptote $u = 1/\beta$ from below.

Quadratic form of (1a). If the travel-time function (1a) is quadratic, $\tau = x + \beta x^2$, then substituting $x = ut$ gives $\tau = x + \beta (ut)^2$ and solving the latter for $\tau$ gives

$$\tau = \frac{1 \pm \sqrt{1 - 4\beta u^2}}{2\beta u^2}$$

(2)

It can be shown that this curve is initially increasing from $(u, \tau) = (0, x)$, bends backward at $(u, \tau) = (1/\sqrt{4\beta}, 2x)$, and continues backward until it becomes asymptotic to the vertical ($\tau$) axis. Thus there is a maximum flow capacity of $u = 1/\sqrt{4\beta}$. To obtain the flow–density function, substitute $\tau = ak/u$ for $\tau$ in the above equation and rearrange, thus,

$$k = \frac{1 - \sqrt{1 - 4\beta u^2}}{2a\beta u}$$

(3)

This does not have an explicit inverse but gives $u$ as an implicit function of $k$. It can be shown that it starts from the origin, increases to a peak flow $u = 1/\sqrt{4\beta}$, then bends downwards and approaches the horizontal ($u$) axis as density $k$ goes to $+\infty$.

We see from the above that the linear and quadratic forms of (1a) have different implications. If (1a) is linear, then travel time as a function of flow, and flow as a function of density, are both monotonic increasing, with no backward bending part. If (1a) is quadratic then these two functions are initially upward sloping and then bend backwards (or downwards).

Though nonlinear forms of (1a) allow backward bending travel-time vs. flow and flow vs. density curves, this is not the cause of the pseudo-periodic effects discussed in this paper. We show
(in propositions and numerical examples) that these effects occur even when traffic is on the upward sloping part of these curves. Also, we do not introduce interactions between links, or queue spill-back onto other links, hence the pseudo-periodic effects obtained in this paper are not cause by such features.

2. Conservation and FIFO properties

We introduce some properties needed below. If the vehicles enter only at the beginning of the link and exit only at the end of the link, that is, no vehicles start or terminate their journeys anywhere along the link, then conservation implies

\[ x(t) = x(0) + \int_0^t u(w) \, dw - \int_0^t v(w) \, dw \]  \hspace{1cm} (4)

where \( u(t) \) and \( v(t) \) are the inflow and outflow rates respectively for the link at time \( t \). If the vehicles also exit in the same order that they enter (i.e. satisfy a first-in-first-out or FIFO condition), then at any time \( t \),

\[ x(0) + \int_{-\infty}^t u(w) \, dw = \int_{-\infty}^{t+s(t)} v(w) \, dw \]  \hspace{1cm} (5)

Differentiating the latter gives \( u(t) = (1 + \tau'(t))v(t + \tau(t)) \), where \( \tau'(t) \) denotes \( d\tau(t)/dt \), and rearranging gives a well-known condition (assuming smooth travel-time functions and continuous inflows),

\[ \tau(t) = \frac{u(t)}{1 + \tau'(t)} \]  \hspace{1cm} (6)

which was discussed in Astarita (1996). Friesz et al. (1993) showed that, under mild conditions, the linear travel-time model (1b) ensures FIFO for traffic on the link, and Wu et al. (1995, 1998) and Xu et al. (1996, 1999) showed this for nonlinear travel-time functions (1a). Any examples that we subsequently consider will satisfy these conditions, hence ensure FIFO, so that (4) and (5) hold.

Eqs. (4)–(6) are well known in the DTA literature and from (1) and (4),

\[ \tau'(t) = \frac{d\tau}{dt} = f''(x(t))[u(t) - v(t)], \]  \hspace{1cm} (7)

where \( f''(x) \) denotes \( df(x)/dx \).

3. Analytic solution for a simple case: quadratic travel-time function and step function inflows

We now find an analytic solution for the model ((1a), (4), (5)) when (1a) is quadratic,

\[ \tau(t) = \alpha + \beta x^2(t) \]  \hspace{1cm} (8)

where \( \alpha \) and \( \beta \) are positive constants. (A quadratic form of the travel-time function has been used in numerical examples in the network literature, for example in Xu et al. (1996, 1999) and Wu et al. (1998).)
Suppose that the inflow, outflow, number of vehicles on the link and travel time were all constant over time (denoted by \( u, v, x \) and \( \tau \) respectively). In that case the model ((1a), (4), (5)) reduces to ((1a), \( u = v, x = ur \)), and if (1a) is also quadratic then the solution is as already been set out in (2) in the introduction. This is a ‘comparative statics’ solutions for given inflows \( u \). However, if inflows change, \( \tau \) does not jump immediately to the new value given by (2). To find how \( \tau \) (and \( v \) and \( x \)) vary over time in response to a change in \( u \) we have to solve ((1a), (4), (5)) dynamically. To illustrate that here, we consider the simplest possible change in \( u \), since even in that case the solution is rather complex. Let the inflows be a step function,**

\[
u(t) = \begin{cases} 0 & t < 0 \\ u_0 & t \geq 0 \end{cases}
\] (9)

and let \( v(0) = 0 \) and \( x(0) = 0 \). The latter implies, from (8), that the initial travel time is \( \tau = x \), hence the traffic \( u_0 \) entering at time \( t = 0 \) exits at time \( \tau = x \), so that there is no traffic outflow until time \( \tau = x \), or more formally

\[
v(t) = 0 \quad 0 \leq t < x
\] (10)

and \( x(t) = u_0t \) when \( 0 \leq t < x \), hence (8) gives

\[
\tau(t) = x + \beta(u_0t)^2 \quad 0 \leq t < x
\] (11)

At time \( x \), outflow commences, with the form given by substituting (9)–(11) into (6) to give

\[
\nu_1(t + x + \beta u_0^2 t^2) = \frac{u_0}{1 + 2\beta u_0^2 t} \quad 0 \leq t < x
\] (12)

and this can be rearranged to give

\[
v(t) = v_1(t) = \frac{u_0}{\sqrt{1 + 4\beta u_0^2 (t - x)}} \quad x \leq t < 2x + \beta u_0^2 x^2
\] (13)

The range for time \( t \) in (13) is derived as follows. Since (6) projects flows \( u(t) \) at time \( t \) into outflows \( v(t + \tau(t)) \) at time \( t + \tau(t) \), the inflows in the interval \( 0 \leq t < x \) become outflows in the interval \( 0 \leq t < x + \tau(x) \). Substituting \( \tau(0) = x \) and (11) into this range gives the range in (13), that is, \( x \leq t < 2x + \beta(u_0x)^2 \).

Note that, from (10), outflow is zero up to time \( t = x \) and, from (13), outflow is \( u_0 \) at time \( t = x \). That is, outflow jumps instantaneously from zero to the inflow value, \( u_0 \), and then gradually falls off as given by (13). Further, over the interval in (13), i.e. \( x \leq t < 2x + \beta u_0^2 x^2 \), the travel time can be found by substituting (13) and (9) into (4) and (8) resulting in,

\[
\tau(t) = \tau_1(t) = x + \beta u_0^2 \left[ t - \frac{1}{2\beta u_0^2} \left( \sqrt{1 + 4\beta u_0^2(t - x)} - 1 \right) \right]^2 \quad x \leq t < 2x + \beta u_0^2 x^2
\] (14)

Now consider inflows over the next time interval, that is, interval to which (13) and (14) apply. From (9) and (6), these inflows yield outflows

\[
v_2(t + \tau_1(t)) = \frac{u_0}{1 + d\tau_1(t)/dt}
\] (15)

Since (6) projects flows \( u(t) \) into outflows \( v(t + \tau(t)) \), the outflows (15) apply to the time interval \( t = x + \tau(x) \) to \( t = x + \tau(x) + \tau(x + \tau(x)) \). We can recast (15) in the form \( v_2(t) = f(t) \) just as we
recast (12) in the form (13). This requires the solution of a quartic equation in \( t \), and although an analytic solution is achievable in principle it is very unwieldy in practice. Further, an analytic solution for the next stage (next time interval) of outflow will prove impossible as it will involve the solution of polynomials in \( t \) of degree greater than 5, thus we stop our analytic solution at this point.

Though finding an analytical solution is impossible, finding a numerical solution for any particular example is relatively straightforward, and is illustrated in Section 5.

4. Pseudo-periodic knock-on effects generated by more general travel-time function

The above section shows that even with a relatively simple form of the travel-time function (1a), namely quadratic, and a simple inflow function, finding an analytic solution is intractable. We now consider more general travel-time functions but, because of the intractability of analytical solution, we derive only the relevant properties of the solutions and not the solutions themselves. This also has the advantage that the results are independent of the form of the travel-time function. For analytical simplicity we assume step changes in inflows, but in a later section will show numerically that similar results hold for continuously changing inflows.

In Section 3 we derived a partial analytic solution for a quadratic travel-time function, and showed that a sudden jump in the inflow rate generates a sequence of jumps and declines in the outflow rate. Here we extend this result to more general travel-time functions, and to consider decreases as well as increases in inflows. For convenience we will assume that the travel-time function \( f(x) \) is twice differentiable with continuous derivatives. However, similar results can be obtained without these assumptions. If the derivatives are discontinuous then, in the following propositions, where we find \( v'(t) > 0 \) (or \( v'(t) < 0 \)) we may also find discontinuous steps increases (or decreases) in \( v(t) \).

**Proposition 1.** Let:

(i) \( u(t) = v(t) = x(t) = 0 \) for \( t < t_0 \), and \( u(t) = u_1 > 0 \) for \( t \geq t_0 \).

(ii) The travel-time function \( \tau = f(x) \) have the properties \( f'(x) = 0 \) when \( x = 0 \); \( f'(x) > 0 \) and \( f''(x) > 0 \) when \( x > 0 \).

(iii) \( f'(x) \leq k \) for all \( x \geq 0 \), where \( k \) is an arbitrarily large finite constant or, more weakly, \( f'(x) \leq k \) for all \( x \) that occur on the link.

Then, as illustrated in Figs. 1 and 2, the outflow rate \( v(t) \):

(a) jumps from 0 to \( u_1 \) at time \( t_1 = \tau(t_0) \),
(b) then declines from time \( t_1 \) until \( t_2 = t_1 + \tau(t_1) \),
(c) then jumps up again to \( u_1 \) at time \( t_2 \),
(d) then declines from time \( t_2 \) until \( t_3 = t_2 + \tau(t_2) \), and so on.

Thus, more generally, for \( n = 1,2,\ldots,+\infty \), the outflow rate \( v(\cdot) \)
(e) jumps up to \( u_1 \) at times \( t_n = t_{n-1} + \tau(t_{n-1}) \), and
(f) then declines from time \( t_n \) to \( t_{n+1} \).
Remark. Assumption (ii) allows, for example, all travel-time functions that are increasing polynomial functions, of the form
\[ s = f(x) = k_0 + k_2x^2 + k_3x^3 + \cdots, \]
with constants \((k_0, k_2) > 0\) and \(k_i \geq 0\) for all integer \(i > 2\). Note, there is no linear term, \(k_1x\).

Assumption (iii), in conjunction with (ii), ensures that \(f(x)\) cannot go to \(+\infty\) at some finite \(x\). If the link travel time \(s = f(x)\) were to go to infinity this would prevent further outflow from the link, hence the number of vehicles on the link would go to infinity.

Proof. (a) At time \(t = t_0\) the inflow rate jumps from 0 to \(u_1\), but this inflow does not reach the exit until time \(^2t_0 + \tau(t_0)\), hence the outflow is zero from time \(t = t_0\) to \(t = \tau(t_0)\). Also, from (i), \(x(t_0) = 0\) hence from (ii), \(f'(x) = 0\) at \(t = t_0\), hence (7) reduces to \(\tau'(t_0) = 0\), hence (6) reduces to \(v(t_0 + \tau(t_0)) = u_1\).

(b) To show that \(v'(t) < 0\) from time \(t_1 = t_0 + \tau(t_0)\) to \(t_2 = t_1 + \tau(t_1)\). Consider the corresponding inflows, i.e. inflows in the interval \(t_0\) to \(t_1\). From (6), \(dv(t + \tau(t))/dt = u'(t)/(1 + \tau'(t)) - u(t)\tau''(t)/\tau'(t)^2\), and we wish to show that this is negative. By assumption

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\(^2\) We let \(\tau'(t)\), \(\tau''(t)\) and \(v'(t)\) denote \(d\tau(t)/dt\), \(d^2\tau(t)/dt^2\) and \(dv(t)/dt\) respectively. Also, for brevity, we let \(\tau'(t_0)\) and \(v'(t_0)\) denote \(\tau(t)\) and \(v(t)\) evaluated at time \(t = t_0\).
(i), \(u(t) = u_1\) hence \(u'(t) = 0\), hence \(dv(t + \tau(t))/dt\) reduces to \(-u(t)\tau''(t)/(1 + \tau'(t))^2\). Also, from (7),

\[
\tau''(t) = f''(x)(u_1 - v(t)) + f'(x)(0 - v'(t))
\]

By assumption (i), for traffic entering from time \(t_0\) to \(t_1\), \(v(t) = 0\) hence \(v'(t) = 0\). This reduces (16a) to \(f''(x)u_1\) and, from assumption (ii), \(f''(x) > 0\) hence (16a) implies \(\tau''(t) > 0\). Combining \(\tau''(t) > 0\) and \(dv(t + \tau(t))/dt = -u(t)\tau''(t)/(1 + \tau'(t))^2\) gives

\[
dv(t + \tau(t))/dt < 0.
\]

Recall that \(dv(t + \tau(t))/dt\) is the derivative with respect to the time \(t\) of entry to the link, whereas what we need is \(d(t')/dr\), the derivative w.r.t. the time of exit \(t' = t + \tau(t)\).

\[
dv(t')/dr = [dv(t + \tau(t))/dt][dt/dr]
\]

To find the sign of \(dt/dr\), note that \(t' = t + \tau(t)\) implies \(dr/dt = 1 + \tau'(t)\) hence \(dt/dr = 1/(1 + \tau'(t))\). But \(\tau'(t) = f'(x)(u_1 - v(t))\) from (7), with \(u_1 > v(t)\) since \(v(t) = 0\) until time \(t_0 + \tau(t_0)\), and \(f'(x) > 0\) by assumption (ii), hence \(\tau'(t) > 0\). Hence \(dr/dr = 1/(1 + \tau'(t))\) implies \(dt/dr > 0\). Hence (16b) and (16c) yield \(dv(t')/dr < 0\). That is, the outflow rate \(v(t)\) declines from time \(t_1 = t_0 + \tau(t_0)\) to time \(t_2 = t_1 + \tau(t_1)\).
Fig. 3. As for Fig. 1 except that \( u(t) = 1.3 \) (exceeds capacity) for \( t \geq 0 \).

Fig. 4. As for Fig. 1 except that \( u(t) = (t/(t+1))^2 \) for \( t \geq 0 \).
(c) To show that \( v(t) \) jumps up to \( u_1 \) at time \( t_2 = t_1 + \tau(t_1) \). From (a), \( v(t) = u_1 \) at time \( t_1 \), hence (7) reduces to \( \tau'(t) = 0 \) at time \( t = t_1 \), hence (6) reduces to \( v(t_1 + \tau(t_1)) = u_1 \) at \( t = t_1 \).

(d) To show that \( v'(t) < 0 \) from time \( t_2 = t_1 + \tau(t_1) \) to \( t_3 = t_2 + \tau(t_2) \). Consider the corresponding inflows, that is, inflows in the interval \( t_1 \) to \( t_2 \). The proof is similar to that for (b) above, but has to be slightly adapted since there we used the fact that \( v_0(t) = 0 \) from time \( t_0 \) to \( t_1 \) while here we use the fact (obtained from (b)) that \( v'(t) < 0 \) from \( t_1 \) to \( t_2 \). Thus we replace the text between (16a) and (16b) with the following:

“Consider the terms in (16a). We saw in (b) that \( v'(t) < 0 \), for traffic entering from time \( t_1 \) to \( t_2 \). Also, from assumption (ii), \( f'(x) > 0 \), hence the second term in (16a) is positive. From assumption (ii), \( f''(x) > 0 \). Also, \( v(t) < u_1 \) (since (a) showed that \( v(t) = u_1 \) at time \( t_1 \) and (b) showed that \( v(t) \) declined thereafter). Combining these reduces (16a) to \( \tau''(t) > 0 \). Then combining \( \tau''(t) > 0 \) and \( dv(t + \tau(t))/dt = -u(t)\tau''(t)/(1 + \tau'(t))^2 \) gives”.

The rest of the proof is as in (b).

(e) Same as proof for (c), but with \( t_2 \) and \( t_1 \) changed to \( t_{n+1} \) and \( t_n \).

(f) Same as proof for (d), but with \( t_2 \), \( t_1 \) and \( t_0 \) changed to \( t_{n+1} \), \( t_n \) and \( t_{n-1} \). □

In Proposition 1 we assumed that the inflow rate is initially zero and jumps from there to a finite value \( u_1 \). In case it might be thought that the initial zero flow is the causes of the knock-on
periodic effects, in the following proposition we instead let the flow initially be in a steady state with a positive value \( u_0 \), and jump from there to a higher value \( u_1 \).

**Proposition 2.** Let:

(i) \( u(t) = v(t) = u_0 > 0 \) for \( t < t_0 \) (hence \( x(t) > 0 \) for \( t < t_0 \)), and \( u(t) = u_1 > u_0 \) for \( t \geq t_0 \).

(ii) \( f'(x) > 0 \) for all \( x > 0 \). Also, \( f''(x) > 0 \) when \( x > 0 \).

(iii) As in Proposition 1.

Then the same results (a)–(f) hold as in Proposition 1, except that in (a), (c) and (d) the outflow \( v(t) \) does not jump up all the way to \( u_1 \) but jumps up to finite values \( u_1 = v(t) = u_1/(1 + ku_1) \), at times: (a) \( t_1 = t_0 + \tau(t_0) \), (c) \( t_2 = t_1 + \tau(t_1) \), and (e) \( t_n = t_{n-1} + \tau(t_{n-1}) \), \( n = 1, 2, \ldots, +\infty \).

**Remark.** This is illustrated in Fig. 6. Assumption (ii) allows for example all travel-time functions that are increasing polynomial functions, of the form \( \tau = f(x) = k_0 + k_1 x + k_2 x^2 + k_3 x^3 + \cdots \), with \( k_0 > 0, k_1 \geq 0, k_2 > 0, \) and \( k_i \geq 0 \) for all integer \( i \geq 2 \). This allows for example quadratic functions \( \tau = k_1 + k_1 x + k_2 x^2 \) or \( \tau = k_1 + k_2 x^2 \), or higher, but not linear functions \( \tau = k_1 + k_1 x \).

![Graph](image-url)  
**Fig. 6.** As for Fig. 1 except that \( u(t) = 0.5 \), instead of 0.0, for \( t < 0 \).
Proof. (a) At time \( t = t_0 \) the inflow rate jumps from \( u_0 \) to \( u_1 \), but this inflow does not reach the exit until time \( t_1 = t_0 + \tau(t_0) \), hence the outflow rate remains at \( v(t) = u_0 \) from time \( t_0 \) to \( t_0 + \tau(t_0) \). Also, from (i), \( x(t_0) > 0 \) hence from (ii), \( f'(x) > 0 \) at \( t = t_0 \), hence (7) reduces to \( \tau(t_0) = f'(x(t_0)) \times \left[ u_1 - v(t_0) \right] < f'(x(t_0))u_1 \) hence from (iii), \( \tau(t_0) < kU_1 \). Hence (6) reduces to \( v(t_0 + \tau(t_0)) \geq u_1/(1 + ku_1) \).

(b) and (d). Similar to that for (b) and (d) in Proposition 1.

(c) To show that \( v(t) \) jumps up to \( u_1 \geq v(t) \geq u_1/(1 + ku_1) \) at time to \( t_2 = t_1 + \tau(t_1) \).

From (7), \( v(t_2) = v(t_1 + \tau(t_1)) = u_1/[1 + f'(t_1)(u_1 - v(t_1))] \geq u_1/[1 + f'(t_1)u_1] \).

Then, using \( k \geq f'(x) \) from assumption (iii), this reduces to \( v(t_2) \geq u_1/(1 + ku_1) \).

(e) The proof is the same as for (c), but with \( t_2 \) and \( t_1 \) changed to \( t_{n+1} \) and \( t_n \). □

Proposition 3. Let:

(i) as for Proposition 2;
(ii) as for Proposition 2, except that \( f''(x) \geq 0 \) for all \( x \geq 0 \), instead of \( f''(x) > 0 \) for all \( x > 0 \);
(iii) as for Propositions 1 and 2.

Then, \( v(t) \) jumps up repeatedly as in Proposition 2 but, instead of declining between jumps, it may remain constant until the next jump up.

Remark. This result is illustrated in figures in Carey and McCartney (2002). Assumptions (ii) allows travel-time functions that are linear (e.g. \( \tau = f(x) = k_0 + k_1x \), with no higher terms), whereas assumption (ii) in Proposition 2 did not.

Proof. Parts (a), (c) and (e) are as in Proposition 2.

(b) We need only consider the case \( f''(x) = 0 \), since the case \( f''(x) > 0 \) was already considered in Proposition 2. Consider (16a) from the proof of part (b) in Proposition 1. If \( f''(x) = 0 \) then the first term in (16a) is zero. Also, \( v(t) \) is constant until inflows \( u_0 \) reach the exit at time \( t = \alpha \), hence \( v'(t) = 0 \) until time \( t = \alpha \). Hence \( v''(t) \) in (16a) reduces to 0, hence (16b) and (16c) reduce to \( dv(t + \tau)/dt = 0 = dv'(\alpha)/dr \), so that the outflows \( v(t) \) remains constant in the next time interval rather than declining.

(d) Consider the proof of (d) from Proposition 2. If \( f''(x) = 0 \) and (as just above) \( v'(t) = 0 \) then again (16a) reduces to \( v''(t) = 0 \), and the rest of the proof is as above. □

In Propositions 1–3 we assumed that the inflow rate jumps up to a new value \( u_0 \) at time \( t = t_0 \). To show that similar results hold for falls in the inflow rate, in the following proposition we instead assume that at time \( t_0 \) the inflows fall from \( u_0 \) to the new value \( u_1 \). The three propositions thus cover all types of discontinuous jumps, whether up or down and whether starting or finishing at zero or at a positive value.
Proposition 4. Let:

(i) as in Proposition 2, but with $u_1 > u_0$ changed to $u_1 < u_0$;
(ii) as in Proposition 2;
(iii) as in Propositions 1–3.

Then, as illustrated in Figs. 7–9, the outflow rate $v(t)$:

(e) drops to values $u_1 \leq v(t) \leq u_1/(1 + ku_1)$ at times $t_1 = t_0 + \tau(t_0)$, $t_2 = t_1 + \tau(t_1), \ldots, t_n = t_{n-1} + \tau(t_{n-1}), \ldots$, and
(f) increases over each interval between these times, e.g. from $t_n$ to $t_{n+1}$, $n = 1, 2, \ldots, +\infty$.

Proof. Same as for Proposition 2, but with appropriate changes of direction of inequalities (caused by changing $u_1 > u_0$ to $u_1 < u_0$) and with “jump to” changed to “drops to” and “decreases” changed to “increases”.

Proposition 5. Let:

(i) as in Proposition 3, but with $u_1 > u_0$ changed to $u_1 < u_0$;
(ii) as in Proposition 3;
(iii) as in Propositions 1–4.

Fig. 7. As for Fig. 1 but inflows have step decrease of 0.9, from $u(t) = 1$ for $t < 0$ to $u(t) = 0.1$ for $t \geq 0$. 
Fig. 8. As for Fig. 7 but inflows have step decrease of 0.5, to \( u(t) = 0.5 \) for \( t \geq 0 \).

Fig. 9. As for Fig. 7 but inflows have step decrease of 0.1, to \( u(t) = 0.9 \) for \( t \geq 0 \).
Then, \( v(t) \) drops repeatedly as in Proposition 4 but, instead of increasing again between jumps, it may remain constant until the next drop.

**Proof.** As for Proposition 4. \( \square \)

In Propositions 2–5 we showed that jumps in inflows generated a series of jumps in outflows, at times \( t_1 = t_0 + \tau(t_0) \), \( t_2 = t_1 + \tau(t_1) \), ..., \( t_n = t_{n-1} + \tau(t_{n-1}) \), ..., converging towards the inflow rate in an infinite series of steps. In the following proposition we characterises the speed of this convergence at each step in the process.

**Proposition 6.** Let:

(i) as in Propositions 2–5, except either \( u_1 > u_0 \) or \( u_1 < u_0 \);
(ii) \( f'(x) \geq 0 \) and \( f''(x) \geq 0 \) for all \( x \geq 0 \),
(iii) as in Propositions 1–5.

Then, after each jump (up or down) in the outflow rate, the remaining percentage difference between the inflow rate \( u_1 \) and the outflow rate \( v(t_{n+1}) \) is proportional to the actual difference (i.e. \( u_1 - v(t_n) \)) just after the previous jump in the outflow rate, and is proportional to the congestion level \( f''(x(t_n)) \) just after the previous jump.

After the initial jump in the outflow rate, at time \( t_1 = t_0 + \tau(t_0) \), the remaining percentage difference between the inflow rate \( u_1 \) and the outflow rate \( v(t) \) is proportional to the initial jump \( (u_1 - u_0) \) in the inflow rate, and is proportional to the initial congestion level \( f''(x(t_0)) \).

**Proof.** Using (7) to substitute for \( \tau'(t) \) in (6), then dividing through by \( u_1 \) and inverting gives, \( u_1/v(t + \tau(t)) = [1 + f'(x)(u_1 - v(t))] \) hence

\[
\frac{u_1 - v(t + \tau(t))}{v(t + \tau(t))} = (u_1 - v(t))f'(x) \tag{17a}
\]

and applying this at time \( t_{n+1} = t_n + \tau(t_n) \) gives

\[
\frac{u_1 - v(t_{n+1})}{v(t_{n+1})} = (u_1 - v(t_n))f''(x(t_n)) \tag{17b}
\]

In particular, applying this at time \( t_1 = t_0 + \tau(t_0) \) gives

\[
\frac{u_1 - v(t_1)}{v(t_1)} = (u_1 - u_0)f'(x(t_0)) \tag{17c}
\]

since by assumption (i), \( v(t_0) = u_0 \). \( \square \)

**Corollary 1.** If there is no congestion at time \( t_n \) (i.e. if \( f'(x(t_n)) = 0 \)) then the outflows adjust fully to inflows at time \( t_{n+1} \).
**Proof.** If \( f'(x(t_n)) = 0 \) then in (17b) and (17c) reduce to \( u_1 = v(t_{n+1}) \) and \( u_1 = v(t) \) respectively. \( \square \)

Part of the above proposition states that, after each jump (up or down) in the outflow rate, the remaining percentage difference between the inflow rate \( u_1 \) and the outflow rate \( v(t) \) is proportional to the congestion level \( f'(x) \). That is intuitively reasonable, since we expect the adjustment of the outflow rate to the inflow rate to be slowed down if congestion is severe. In particular, the corollary states that if there is no congestion then the flow rate does not change as traffic moves along the link, so that the outflow rate when a vehicle exits is equal to the inflow rate when it entered.

5. Numerical examples

In Section 3 we demonstrated that even with a relatively simple form of travel-time function, namely quadratic, and the simplest form of inflows (a step function) finding an analytic solution was possible only for the first few time steps after a step change in the inflow rate. In view of that, in Section 4 we considered only general properties of solutions. However, though solving analytically is not possible, solving numerically is relatively easy. In view of that, in what follows we present and discuss numerical results that illustrate the theoretical results from Sections 3 and 4. These results are generated by numerically solving the quadratic model described by Eqs. (8), (4) and (6). The examples below do not use any results from Section 3 or 4, hence they independently illustrate the results from Sections 3 and 4.

These results are generated by numerically solving the quadratic model described by Eqs. (8), (4) and (6). The examples below do not use any results from Section 3 or 4, hence they independently illustrate the results from Sections 3 and 4. To solve the model, in each example we used a fine discretisation of time, namely time steps of 0.1, and to ensure that the discretisation was sufficiently fine, we checked that the results were not significantly affected by using a discretisation 0.01 and 0.001.

Fig. 1 illustrates the solution for a step function inflow of the form (9). Note the sequence of discontinuities in the outflow. These discontinuities occur at times \( T_n \) where

\[
T_n = T_{n-1} + \tau(T_{n-1}) \quad \text{for } n = 0, 1, 2, 3, \ldots, \infty
\]

with the first discontinuity occurring at \( T_0 = x \). At each discontinuity the outflow jumps to the value of the constant inflow, \( u_0 \). The parameters for Fig. 1 were chosen so that \( u < 1/\sqrt{4\alpha\beta} \), the latter being the capacity limit that is derived just after Eq. (2) in the introduction and is the maximum constant flow rate that is feasible with a quadratic travel-time function (8). Keeping inflow less than or equal to the capacity limit allows the flow on the link to stabilise to the steady state with inflow = outflow, so as time increases we observe that the discontinuities die out with outflow approaching the constant inflow value, \( \lim_{t \to \infty} v(t) = u_0 \). This implies \( \lim_{t \to \infty} x(t) = \lim_{t \to \infty} u_0 \tau(t) \) so that from (8), in the limit, \( \tau(t) = x + \beta u_0^2 \tau^2(t) \) and solving this quadratic, the travel time on the link approaches

\[
\lim_{t \to \infty} \tau(t) = \frac{1 - \sqrt{1 - 4\alpha\beta u_0^2}}{2\beta u_0^2}
\]

and, from \( \lim_{t \to \infty} x(t) = \lim_{t \to \infty} u_0 \tau(t) \), the number of vehicles on the link approaches

\[
\lim_{t \to \infty} x(t) = \frac{1 - \sqrt{1 - 4\alpha\beta u_0^2}}{2\beta u_0}
\]
These limits are also the solution values obtained in (2) and (3) in the introduction for the case in which the flows and link load \((u, v\) and \(x\)) are constant over all time. The above is illustrated in Fig. 2 where we extend the results from Fig. 1 over a much longer time span.

Fig. 3 illustrates the case where \(u_1\) exceeds the stable link capacity \(1/\sqrt{4\alpha\beta}\). In this case there is no stable limit for \(v(t)\) as \(t \to \infty\), and instead of the discontinuities damping out over time they increase in severity. This is a significant difference between the linear and quadratic travel-time models. Earlier work (Carey and McCartney, 2002) on the linear travel-time model has shown that for constant inflow the severity of the discontinuities in the outflow will always die out over time. The graph of link travel times \(s(t)\) appears smooth. However, the graph of \(ds/dt\) (not shown) is upward slopping but with downward spikes similar to those in the graph of outflows but of shorter duration. Because these spikes in \(ds/dt\) are of such short duration they do not show up on the graph of \(s(t)\).

In traffic flow theory, what happens in Fig. 3 can be explained by considering the flow–density function corresponding to the quadratic travel-time function (8). The inflow rate in this example has been chosen so as to exceed the link capacity (the peak of the flow–density function) hence the number of vehicles on the link has moved onto the downward slopping part of the flow–density curve. As the flow continues at above capacity level, the flow and density move down the downward slopping part of the flow–density curve, hence the increasing \(x\) and decreasing outflow \(u\) in Fig. 3.

It might be assumed that the behaviour illustrated in the Figs. 1–3 results principally from the presence of a finite jump discontinuity in the inflow pattern. That this is not the case is illustrated by Figs. 4 and 5 where an inflow of the form

\[
  u(t) = 0 \quad \text{when } t < 0 \quad \text{and} \quad u(t) = (t/(t + \gamma))^2 \quad \text{when } t \geq 0
\]  

(21)

is considered. The derivative of (21) is

\[
  du(t)/dt = 0 \quad \text{when } t < 0 \quad \text{and} \quad du(t)/dt = 2\gamma/(t + \gamma)^3 \quad \text{when } t \geq 0
\]  

(22)

and since this is zero at \(t = 0\), not only is \(u(t)\) continuous but its derivative is continuous. Since (22) is increasing, has the effect of decreasing the sharpness of increase of \(u(t)\). Fig. 4 shows a rapid, but smooth, increase in inflow \((\gamma = 1)\) giving rise to set of discontinuities in outflow. In Fig. 5 we increase the value of \(\gamma\) in the inflow function to 10 and the discontinuities in outflows effectively disappear.

In Figs. 1–5 it is assumed that the initial inflows, outflows and traffic on the link are all zero, i.e. \(u_0 = v(0) = x(0) = 0\). In case it may be thought that this explains the nature of the results, in Fig. 6 we instead assume that, up to time zero, the inflows and outflows are at a positive level \((u_0 > 0)\), and that \(x(0)\) is at the level implied by constant inflow = outflows = \(u_0\). The scenario is otherwise as in Fig. 1. The results (Fig. 6) are similar to Fig. 1, except that each time the outflows jump up they do not jump all the way up to the inflow level \(u_0\). (This difference is also shown in Propositions 1 and 2.)

In Figs. 1–6 we assume initial increases in inflow rates, and show the knock-on effect of these on the outflow rates. In Figs. 7–9 we show that similar results obtain if instead there is an initial decrease in inflow rates. We show three figures (Figs. 7–9) instead of just one, since the differences
between the figures are somewhat surprisingly. In the figures with the smaller initial fall in inflows, the knock-on effect of this on outflows is visible (non-negligible) for a much longer span of time. This is a real effect and not just a graphical scale effect, and is true whether we consider the change in inflows in absolute terms or in relative (percentage) terms. A fall of 0.9 (or 90%) in inflows in Fig. 7 has a significant effect on outflows until time 50, while a smaller fall of 0.5 (or 50%) in Fig. 8 has a significant effect on outflows until time 70; and a fall of only 0.1 (or 10%) in Fig. 9 has a significant effect until time 100.

An explanation for this phenomenon is provided in Appendix A. There we consider an initial fall in inflows (as in Figs. 7–9), which causes a series of falls in the outflow rate, with outflows converging to the new inflow rate. We show that at the $n$th jump in the outflows the remaining (normalised) gap between the outflow rate and the inflow rate is proportional to $(u_1/u_0)^n$. Since $u_1/u_0 < 1$, the smaller is $u_1/u_0$ the faster is convergence. That is, if there is a large initial change in inflows, then outflows converge to this very quickly and, conversely, if there is only a small initial change in inflows then outflows may take a long time to converge to this. Note that this holds even if we start from the same initial inflow rate $u_0$ in each case, and use $u_0$ to normalise the outflows and convergence gap. This result is counterintuitive, since it seems natural to expect that the larger the (normalised) jump in inflows the longer outflows would take to converge to this.

Perhaps equally surprising is that the above counter intuitive phenomenon applies if there is a decrease in inflows but may not apply if there is an increase, as is shown in the later part of Appendix A and can be confirmed by comparing Figs. 6 and 10. In Fig. 10(a) the jump in inflow is

Fig. 10. As for Fig. 6 except inflows jump by 0.1 instead of 0.5, from $u(t) = 0.9$ for $t < 0$ to $u(t) = 1.0$ for $t \geq 0$. 
smaller than in Fig. 6(a) (0.1 instead of 0.5) and it can be seen (they are drawn to the same scale) that the convergence of outflow to inflow is somewhat faster. In the analysis in Appendix A we assumed, for simplicity, a quadratic travel-time function. However, similar results may be obtained using more general travel-time functions.

6. Concluding remarks

In this paper we considered the link travel-time model (1) and some of its implications for travel times and outflow rates when flows are changing over time. We show that if there is a sharp change (increase or decrease) in the inflow rate this generates a sharp change in the outflow rate. This sharp change in outflows in turn generates an infinite series of sharp changes in the outflow rate, which damp out over time (unless we apply inflows that exceed the link capacity).

To more dramatically demonstrate this pseudo-periodic knock on effect we focussed on sharp or “rapid” changes in inflows, that is, a large change in inflows over a time span shorter than the time taken to traverse the link. However, the knock-on effects are still present, though less noticeable, even when there are much slower or milder changes in the inflow rate. In that case the knock-on effects tend to be smoothed out and submerged in other changes.

We should also note that the above pseudo-periodic effect does not reflect a real traffic phenomenon, but is an “accidental” by-product of (1). It arises because the link travel-time (1) is a function only of the number of vehicles on the link. This means that when traffic exits from the link this immediately reduces the number of vehicles on the link, which (from (1)) immediately reduces the link travel time. In reality, especially for long links, we would not expect that if some traffic exits at one end of a link this would immediately affect the travel time for new traffic entering at the other end.

On the other hand, if traffic flows are not changing very sharply over time, then the model (1) may be a very acceptable approximation. Note that if traffic inflow is building up, or falling off, over time then (1) also introduces a (short) time lead or lag in predicting the outflow. This is because (1) assumes that the travel time for each vehicle depends on the number of vehicles on the whole link at the snapshot instant \( t \) when the vehicle enters the link, rather than depending on the traffic flow experienced by the vehicle as it traverses the link after time \( t \).

What are some implications of the pseudo-periodic effect investigated in this paper, for using the travel-time model ((1), (4), (5)) in practice with time-varying flows? First, it suggests that we should be wary of applying the model to situations where there are sudden or rapid increases in inflows, or to model change in inflows that occur over a time span that is shorter than the time taken to traverse the link. Thus, for example, the model should not be used to model the rapid changes that follow traffic controls or traffic lights. That will not be a surprise to those who have proposed using the model, and indeed there are also other reasons for not applying the model in such situations. Paradoxically, the model is better at handling sudden or rapid falls in inflows than increases in inflows, as explained in the last few paragraphs of Section 5. However, in a network context it is likely that if sudden or rapid falls in inflows would occur then sudden or rapid increases would also occur. Second, if we use the model in a network assignment model and if rapid endogenous changes in inflows or outflows should happen to occur on some links, then we should bear in mind the above pseudo-periodic phenomenon when interpreting the solution (the outflow
and travel time profiles). Further, when solving the model, the time discretisation should be sufficiently fine as to ensure that the discrete differences between the link outflows in successive time steps are relatively small. These discrete changes in outflow can generate a pseudo-periodic series of further changes in outflows, but if these are sufficiently small they will be negligible and can be ignored.

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Appendix A. The effect of jump discontinuities in inflow for the quadratic travel-time model

This appendix proves results referred to in discussion of Figs. 7–10 in Section 5, concerning the speed of convergence of outflows to a new level after a discontinuous change in inflows. The scenario considered is as follows. Consider a stable flow $u_0$ on a link governed by the model ((1a), (4)–(6)) where (1a) is quadratic as in (8), that is

$$\tau(t) = \alpha + \beta x^2(t)$$

(A.1)

where $\alpha$ and $\beta$ are positive constants. By stable flow we mean that the link outflow $v(t)$, and the number of vehicles on the link $x(t)$, have adjusted to values that are constant over time. From (4), $dx/dt = u(t) - v(t)$ and for stable flow $dx/dt = 0$, hence $v(t) = u(t) = u_0$. Adding (4) and (5) gives $x(t) = \int_{t-\tau}^{t} v(w) \, dw$ and, since $v(t) = u_0$ this reduces to $x = \tau u_0$, which reduces (A.1) to $\tau = x + \beta (ut)^2$. The solution of the latter, hence the solution of (A.1) under stable conditions, is

$$T_{\pm} = \frac{1 \pm \sqrt{1 - 4\alpha \beta u_0^2}}{2\beta u_0^2}$$

(A.2)

$T_-$ corresponds to the travel time for uncongested stable flow, i.e. on the upward sloping part of the travel-time vs. flow rate curve, $\tau = h(u)$.

A.1. Decrease in stable flow

Consider a link with stable flow, as above, and inflow $u_0$ up to time $t = 0$, and at time $t = 0$ let the inflow drops to $u_1 < u_0$ and remains there, i.e.

$$u(t) = \begin{cases} u_0 & t < 0 \\ u_1 & (t < u_0) & t \geq 0 \end{cases}$$

(A.3)

The outflow is governed by, from (6) and (A.1), and recalling that (4) implies $dx/dt = u(t) - v(t)$,

$$v(t + \tau(t)) = \frac{u(t)}{1 + 2\beta(u(t) - v(t))x(t)}$$
and, as seen in Sections 3 and 4, the discontinuity in \( u(t) \) will result in a set of discontinuities in \( v(t) \) occurring at times \( T_1, T_2, T_3, \ldots \), with the value of \( v(t) \) immediately after the \( n \)th change being given by

\[
v_n = \frac{u_1}{1 + 2\beta(u_1 - v_{n-1})x_{n-1}}
\]  
(A.4)

where \( x_{n-1} \) is the number of vehicles on the link at the time of the \((n-1)\)th change. To measure convergence of this outflow \( v_n \) to the inflow \( u_1 \), we scale the difference in terms of the given initial flow \( u_0 \) thus, from (A.4),

\[
\frac{v_n - u_1}{u_0} = \left( \frac{u_1}{u_0} \right) \frac{2\beta(v_{n-1} - u_1)x_{n-1}}{1 - 2\beta(v_{n-1} - u_1)x_{n-1}}
\]  
(A.5)

For ease of manipulation we write

\[
R_n = \frac{v_n - u_1}{u_0}
\]  
(A.6)

\[
B = 2\beta u_0, \quad A = \left( \frac{u_1}{u_0} \right) B
\]  
(A.7)

and so the convergence measure (A.5) can be rewritten as

\[
R_n = \frac{AR_{n-1}x_{n-1}}{1 - BR_{n-1}x_{n-1}}
\]  
(A.8)

Given that initially \( v = u_0 \) then from (A.6)

\[
R_0 = \frac{u_0 - u_1}{u_0}
\]  
(A.9)

Using (A.9) in (A.8) we can generate expressions for \( R_1, R_2 \) and \( R_3 \)

\[
R_1 = \frac{AR_0x_0}{1 - BR_0x_0}
\]

\[
R_2 = \frac{AR_1x_1}{1 - BR_1x_1} = \frac{A^2R_0x_0x_1}{1 - BR_0(x_0 + A_0x_1)}
\]  
(A.10)

\[
R_3 = \frac{AR_2x_2}{1 - BR_2x_2} = \frac{A^3R_0x_0x_1x_2}{1 - BR_0(x_0 + A_0x_1 + A^2_0x_1x_2)}
\]

From the form of Eq. (A.10) we guess that the general form of \( R_n \) is given by

\[
R_n = \frac{A^nR_0 \prod_{j=0}^{n-1} x_j}{1 - BR_0 \sum_{i=0}^{n-1} A^i \prod_{j=0}^{i} x_j}
\]  
(A.11)

That expression (A.11) is in fact correct can easily be proved by induction. From (A.2) the stable number of vehicles on the link at \( t = 0 \) is given by \( x_0 = u_0T_0 = (1 - \sqrt{1 - 4\beta u_0^2})/2\beta u_0^2 \). From Eq. (A.7) we can then write \( B = (1 - \sqrt{1 - 4\beta u_0^2})/x_0 \) and using this and (A.6) and (A.7) in (A.11) gives the convergence measure.
A.2. Increase in stable flow

Next, consider the case in which such a stable flow exists and then at \( t = 0 \) the inflow on the link increases to a value \( u_1 > u_0 \), i.e.

\[
u(t) = \begin{cases} u_0 & t < 0 \\ u_1 (> u_0) & t \geq 0 \end{cases}
\]

(A.15)
Solving in exactly the same manner as in the previous section gives

\[
R_n = \frac{\left(\frac{u_1 - u_0}{u_0}\right) f_n \prod_{j=0}^{n-1} \left(\frac{x_j}{x_\infty}\right)}{1 + \left(\frac{u_0}{u_1}\right) \left(\frac{u_1 - u_0}{u_0}\right) \sum_{i=0}^{n-1} f_i \prod_{j=0}^{i+1} \left(\frac{x_j}{x_\infty}\right)}
\]  

(A.16)

where

\[
R_n = \frac{u_1 - v_x}{u_0} \quad x_\infty = \frac{1 - \sqrt{1 - 4\beta u_1^2}}{2\beta u_1} \quad f_\infty = 1 - \sqrt{1 - 4\beta u_1^2}
\]

(A.17)

Note that \(x_\infty\) is the number of vehicles on the link corresponding to a stable flow of \(u_1\). That is, it is the number of vehicles which the solution approaches asymptotically. Further, note that

\[0 \leq \frac{u_0}{u_1}, f_\infty, \frac{x_j}{x_\infty} \leq 1.\]

The solution (A.16) for a finite jump in inflow is similar to the solution (A.12) for a finite drop in inflow. One important difference however should be noted, namely (A.16) does not retain the strong dependence on the ratio of old to new inflow, \(u_0/u_1\). As shown earlier, this dependence caused the counterintuitive effect that the effects of small drop discontinuities take longer to die out from the resultant outflow pattern than the effects of large drop discontinuities. Thus, we would expect this counterintuitive effect to be to be much less prominent, if present at all, in the outflow pattern if a discontinuous increase occurs in the inflow pattern. That this is indeed the case is illustrated by comparing Figs. 6 and 10.

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