The existence, uniqueness and computation of an arc-based dynamic network user equilibrium formulation

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Abstract

In this paper, a dynamic user equilibrium traffic assignment model with simultaneous departure time/route choices and elastic demands is formulated as an arc-based nonlinear complementarity problem on congested traffic networks. The four objectives of this paper are (1) to develop an arc-based formulation which obviates the use of path-specific variables, (2) to establish existence of a dynamic user equilibrium solution to the model using Brouwer’s fixed-point theorem, (3) to show that the vectors of total arc inflows and associated minimum unit travel costs are unique by imposing strict monotonicity conditions on the arc travel cost and demand functions along with a smoothness condition on the equilibria, and (4) to develop a heuristic algorithm that requires neither a path enumeration nor a storage of path-specific flow and cost information. Computational results are presented for a simple test network with 4 arcs, 3 nodes, and 2 origin–destination pairs over the time interval of 120 periods.

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1. Introduction

The prediction of time-varying flow and cost patterns on congested traffic networks, referred to as dynamic traffic assignment, is considered one of the most challenging tasks among transportation researchers. A comprehensive review of various dynamic traffic assignment models can be
found in Ran and Boyce (1996). This paper is mainly concerned with a particular class of dynamic traffic assignment problem, that is, a user equilibrium simultaneous departure time and route choice problem with elastic demands. A similar problem with fixed demands has been studied in the works of Friesz et al. (1993) and Wie et al. (1995a,b) that present the route-based variational inequality models and algorithms and in the work of Ran et al. (1996) that presents an arc-based variational inequality model.

The motivation of formulating an arc-based model in this paper is to reduce the modeling complexity and computational difficulty by avoiding the use of path-specific flow variables and cost functions. The four objectives of this paper are (1) to develop an arc-based formulation which obviates the use of path-specific variables, (2) to establish existence of a dynamic user equilibrium solution to the nonlinear complementarity model using Brouwer’s fixed-point theorem, (3) to show that the time-varying total arc flow and associated minimum cost patterns are unique by imposing strict monotonicity conditions on the arc travel cost and demand functions along with a smoothness condition on the equilibria, and (4) to develop a heuristic algorithm that requires neither a path enumeration nor a storage of path-specific flow and cost information.

It should be noted that our arc-based approach to formulate a dynamic user equilibrium model is a significant departure from the path-based formulation used in Friesz et al. (1993). It is not trivial how to handle the arc dynamics in discrete time in a manner in which the outflows and the arc travel times are consistent. Hence, both the formulation of a dynamic user equilibrium model using an arc-based approach and the discrete-time version of the arc dynamics providing outflows that are consistent with arc travel times are significant contributions. It should also be noted that the continuous-time version of our arc-based formulation is new and a significant departure from continuous-time models considered to date.

Our arc-based model is developed in discrete time not only to avoid the complexity of infinite-dimensional mathematical analysis of a continuous-time model, but also to take advantage of the simplicity of discrete-time numerical analysis. Although it can be developed as a continuous-time model in the first place, a continuous-time model needs to be eventually discretized to develop a numerical method. Even if a continuous-time model has certain desirable properties of existence or uniqueness, it does not automatically follow that these carry over to a discrete-time version or analogy. These properties need to be derived or proved again for a discrete-time version, and the method of proof is different.

The remainder of this paper is organized as follows. In Section 2, a dynamic network user equilibrium problem with elastic demands is formulated which uses arc-based variables. It is then shown in Section 3 that the problem is equivalent to an arc-based nonlinear complementarity problem. In Section 4, using Brouwer’s fixed-point theorem, we establish existence of a dynamic user equilibrium solution to the nonlinear complementarity problem. In Section 5, by imposing strict monotonicity conditions on the arc travel cost and demand functions along with a smoothness condition on the equilibria, we show that time-varying travel costs and aggregate arc flow patterns are unique. In Section 6, a heuristic algorithm is described with details of each iterative step. In Section 7, the proposed algorithm is implemented on a simple test network with 4 arcs, 3 nodes, and 2 origin–destination pairs over the time interval of 120 periods. The paper concludes with a discussion on future extensions of the proposed model and algorithm.
2. Dynamic network user equilibrium model

In this section, we formulate an arc-based discrete time dynamic network user equilibrium traffic assignment model with elastic demands. Let $G(M,A)$ denote an abstract transportation network comprised of a finite set of nodes, $M$, and a finite set of directed arcs, $A$. The network is said to be complete if there is a directed path from every node to every other node. Each commuter travels from a specific origin $k \in K \subseteq M$ to a specific destination $n \in N \subseteq M$ on some path $p$ that is an acyclic chain of arcs. The index $a$ will denote an arc, the index $k$ an origin node, the index $l$ an intersection (or junction) node, and the index $n$ a destination node. In particular, the index $l$ will denote a head node of arc $a$ represented as either $(k,l)$ or $(j,l)$. Let $P_{kn}$ denote the set of all possible paths between any $k \in K$ and $n \in N$. The set of all possible paths in the network is denoted by $P$.

The time interval of analysis is discretized to create a finite set of time periods of uniformly small length $\Delta$, $T = \{t | t = 0, 1, \ldots, T\}$. Each time period indexed by an integer $t$ represents the $t$th period $[t\Delta, (t+1)\Delta)$. Let $\gamma_a$ denote the free-flow travel time on arc $a$. We assume that the length of each time period is shorter than the minimum free-flow arc travel time; that is, $\Delta < \gamma_{\min}$, where $\gamma_{\min} = \min \{\gamma_a : a \in A\}$. This means that no vehicle can be present on more than one arc during the same time period. We also assume that the time interval of analysis is long enough so that no vehicle will remain on the network in the terminal time period $T$. Note that traffic volume is defined as the number of vehicles present on an arc at some instant in time. Traffic volume may be used to compute traffic density as a measure of vehicle concentration per unit length of an arc. Also note that traffic flow rate is defined as the number of vehicles per unit time passing a fixed point (either an entry point or an exit point) on an arc, and traffic flow is defined as the product of traffic flow rate by length of time period, $\Delta$. We list the notation of key variables that will be employed in the remainder of this paper:

- $v_{ant}$: the rate of vehicles originating at some instant in period $t$ at the tail node of arc $a$, entering arc $a$, and traveling to node $n$, referred to as the departure arc inflow rate.
- $w_{ant}$: the rate of vehicles exiting from upstream (i.e., preceding) arcs whose head nodes are the tail node of arc $a$ at some instant in period $t$, entering arc $a$, and traveling to node $n$, referred to as the transient arc inflow rate.
- $v$: [\(v_{ant} : a \in A, n \in N, t \in T\)]
- $w$: [\(w_{ant} : a \in A, n \in N, t \in T\)]
- $r_{ant}(v,w)$: the rate of vehicles exiting from arc $a$ at some instant in period $t$ and traveling to node $n$, referred to as the arc outflow rate.
- $u_{ant}$: the rate of vehicles entering arc $a$ at some instant in period $t$ and traveling to node $n$, referred to as the arc inflow rate.
- $x_{ant}$: the number of vehicles present on arc $a$ at the beginning of time period $t$ (i.e., at time $t\Delta$) which are destined to node $n$, referred to as the arc traffic volume.

Our proposed approach to model arc traffic dynamics and flow propagation mechanism only is similar to the approach based on the use of arc exit time functions and their inverses. This approach is originally proposed by Friesz et al. (1993) and extended by Astarita (1996), Wu et al. (1998) and Xu et al. (1999). The advantage of this approach is to ensure internal consistencies among arc traffic dynamics, flow propagation constraints and arc delay functions under plausible
regularity conditions. Although the intrinsic complexity of this approach stems from the use of arc exit time functions and their inverses, recent numerical advances reported by Wu et al. (1998) and Xu et al. (1999) demonstrate that its potential application to large-scale networks appears promising. The major difference is that we use point-to-set mappings in discrete time instead of arc exit time functions and their inverses in continuous time.

Let $C_a(x_{at}, u_{at})$ denote the actual time taken to traverse arc $a$ for vehicles entering it in period $t$ when the total number of vehicles present on arc $a$ at the beginning of time period $t$ is $x_{at}$ and the rate of vehicles entering arc $a$ in period $t$ is $u_{at}$. As will be discussed subsequently in this paper, it is more convenient to let $D_{at}(v, w)$ denote the actual time to traverse arc $a$ when entering in period $t$ under traffic conditions $(v, w)$. We assume that the arc travel time is the same for all traffic that enters the arc in the same time period regardless of their destinations. We also assume that $D_{at}(v, w)$ is positive for all arcs in all time periods. Note that $D_{at}(v, w)$ is the value of $C_a(x_{at}, u_{at})$ when the network is loaded according to $(v, w)$ following traffic dynamics and flow propagation mechanism specified subsequently in this section.

In the continuous-time models of Friesz et al. (1993), Astarita (1996), Wu et al. (1998), and Xu et al. (1999), the arc travel time for vehicles entering the arc at time $t$ is treated as a function of the number of vehicles present on the arc at time $t$. However, because we are working in discrete time, we must include the arc inflow in the travel time function to capture the average number of vehicles on the arc during each time period. Otherwise, it would be possible in a particular time period to load the network heavily without the delay that reflects the traffic volume accumulated during that time period. If there were no effect in time period $t$ of very large inflows, the true average delay during that period would not be taken into consideration. This causes the numerical algorithms to be less stable. Essentially, the arc travel time function used in our model can capture arc capacity constraints and point queues through the selection of an appropriate functional form. However, it cannot handle any interactions among separate arcs such as spill-back of queues from congested arcs to the upstream arcs. The reader is referred to the work of Adamo et al. (1999) for handling the spill-back of congestion in a dynamic network simulation model.

The point-to-point mapping of entrance time periods to exit time periods is defined as

$$ z'_a = \{ \epsilon | \epsilon A \leq tA + D_{at}(v, w) < (\epsilon + 1)A \} \quad \forall a \in A, \ t \in T. \quad (1) $$

Note that with this mapping, for any $t \in T$, there is a unique $\epsilon$ which is the time period of exit from arc $a$ if entrance to arc $a$ occurs in period $t$; that is, $|z'_a| = 1$ for all $a \in A$ and $t \in T$. For simplicity, $z'_a$ will also be used to denote the time period of exit from arc $a$ if entrance to arc $a$ occurs in period $t$. The point-to-set mapping of arc exit time periods to arc entrance time periods is defined as

$$ \theta'_a = \{ \pi | tA \leq \pi A + D_{at}(v, w) < (t + 1)A \} \quad \forall a \in A, \ t \in T. \quad (2) $$

Note that the mapping of arc exit time periods to arc entrance time periods can map a single exit time period into consecutive entrance time periods because arc delays can be decreasing; that is, $|\theta'_a| \geq 1$ for all $a \in A$ and $t \in T$. For each $\pi \in \theta'_a$, define

$$ \rho_{at} = \frac{\pi A + D_{at}(v, w) - tA}{A} \quad \forall a \in A, \ t \in T, \quad (3) $$

where $0 \leq \rho_{at} < 1$.

The dynamics of destination-specific traffic on each arc is described by the following difference equation with the initial condition:
\[ x_{an,t+1} - x_{an_t} = [v_{an_t} + w_{an_t} - r_{an}(v,w)] \Delta \quad \forall a \in A, \, n \in N, \, t \in T, \quad (4) \]
\[ x_{an_0} = 0 \quad \forall a \in A, \, n \in N, \quad (5) \]

Note that \( v_{an_0} \) is the flow rate of vehicles that, in period \( t \), are just starting their trips from the tail node of arc \( a \), entering arc \( a \) and traveling to node \( n \), \( w_{an_0} \) is the flow rate of vehicles that, in period \( t \), are just passing through the tail node of arc \( a \) from upstream arcs, entering arc \( a \) and traveling to node \( n \), and \( r_{an}(v,w) \) is the flow rate of vehicles that, in period \( t \), are exiting from the head node of arc \( a \) and traveling to node \( n \). Also note that \( [v_{an_t} + w_{an_t}] \Delta \) is the number of vehicles with destination \( n \) that have entered arc \( a \) during the time interval of \([t \Delta, (t + 1) \Delta)\) and \( r_{an}(v,w) \Delta \) is the number of vehicles with destination \( n \) that have exited from arc \( a \) during the time interval of \([t \Delta, (t + 1) \Delta)\). In the remainder of this paper, \( v_{an_t} \) will be referred to as the departure inflow, \( w_{an_t} \) as the transient inflow, and \( r_{an}(v,w) \Delta \) as the outflow. It can be shown that \( r_{an}(v,w) \) varies continuously in \( v \) and \( w \) and \( x_{an_0} \) is continuous in \( v \) and \( w \), assuming that \( D_{n}(v,w) \) is also continuous in \( v \) and \( w \).

In the above arc traffic dynamics (4)–(6), we assume that a platoon (or packet) of vehicles with different destinations is formed at the tail node of arc \( a \) at the time of entrance and remains intact until the time of exit from the same arc. That is, \( v_{an_t} \Delta \) and \( w_{an_t} \Delta \) are platoons of vehicles with different departure time choices and routing decisions. We also assume that each platoon may be divided into smaller platoons at the head node of arc \( a \) at the time of exit; that is, a new set of platoons may be formed at any intersection node before entering downstream arcs. The first-in-first-out (FIFO) queue discipline is satisfied on each arc if the following regularity condition holds: if \( t' < t'' \), \( t' \in T \) and \( t'' \in T \):
\[ (t'' - t')[t'' \Delta + D_{ar'}(v,w) - t' \Delta - D_{ar}(v,w)] > 0 \quad \forall a \in A. \quad (7) \]

Note that the above FIFO regularity condition implies that “overtaking” is not allowed on each arc.

We make the following definition:
\[ u_{an_t} = (v_{an_t} + w_{an_t}) \Delta \quad \forall a \in A, \, n \in N, \, t \in T. \quad (8) \]

To relate the flow rates \([v_{at}, w_{at}, u_{at}, r_{at}(v,w)]\) and the volumes \( x_{at} \) to their components we need:
\[ v_{at} = \sum_{n \in N} v_{an_t} \quad \forall a \in A, \, t \in T, \quad (9) \]
\[ w_{at} = \sum_{n \in N} w_{an_t} \quad \forall a \in A, \, t \in T, \quad (10) \]
\[ u_{at} = \sum_{n \in N} u_{an_t} \quad \forall a \in A, \, t \in T, \quad (11) \]
\[ r_{at}(v,w) = \sum_{n \in N} r_{an}(v,w) \quad \forall a \in A, \, t \in T, \quad (12) \]
\[ x_{at} = \sum_{n \in N} x_{an_t} \quad \forall a \in A, \, t \in T. \quad (13) \]
In the rest of this paper, \( u_{at} \Delta \) will be referred to as the total arc inflow, \( r_{at}(v,w) \Delta \) as the total arc outflow, and \( x_{at} \) as the total arc volume. Note that aggregate forms of the arc traffic dynamics in (4)–(6), without the \( n \) subscripts, can be obtained by summing (4), (5) and (6), respectively over all \( n \in N \) and substituting from (8)–(13).

In the formulations that follow, we assume that \( D_{at}(v,w) \) is evaluated by loading the network according to \( (v,w) \) and (4)–(6). Loading the network makes the arc traffic volumes \( x_{at} \) available and \( D_{at}(v,w) \) is set equal to \( C_a(x_{at}, u_{at}) \). To better understand this dynamic network loading procedure, we need to define the following variables:

\[
V_{a,t-1} = \sum_{s=0}^{t-1} v_{as} \Delta \quad \forall a \in A, \ t \in T, \quad (14)
\]

\[
W_{a,t-1} = \sum_{s=0}^{t-1} w_{as} \Delta \quad \forall a \in A, \ t \in T, \quad (15)
\]

\[
R_{a,t-1} = \sum_{s=0}^{t-1} r_{as}(v,w) \Delta \quad \forall a \in A, \ t \in T, \quad (16)
\]

where \( V_{a,t-1} \) is the cumulative departure inflow on arc \( a \) during the time interval \([0,t\Delta]\), \( W_{a,t-1} \) is the cumulative transient inflow on arc \( a \) during the time interval \([0,t\Delta]\), and \( R_{a,t-1} \) is the cumulative outflow on arc \( a \) during the time interval \([0,t\Delta]\). The traffic volumes \( x_{at} \) can be expressed as

\[
x_{at} = V_{a,t-1} + W_{a,t-1} - R_{a,t-1} \quad \forall a \in A, \ t \in T. \quad (17)
\]

Therefore, the variables \( x_{at} \) do not appear explicitly in the formulations presented in the rest of this paper.

Consider an arbitrary path \( p \in P_{kn} \) connecting the origin \( k \) and the destination \( n \) and express it as

\[
p = [k = k_0, a_1, k_1, \ldots, k_{m_p-1}, a_{m_p}, n = k_{m_p}]. \quad (18)
\]

The point-to-point mapping of entrance time periods to exit time periods along a path \( p \) can be defined as

\[
z'_{a_i} = \{ \epsilon | \epsilon \Delta \leq t \Delta + D_{a_i}(v,w) < (\epsilon + 1)\Delta \}, \quad (19)
\]

\[
z_{a_i} = \{ \epsilon | \epsilon \Delta \leq t \Delta + D_{a_i}(v,w) + \cdots + D_{a_{i-1}}(v,w) < (\epsilon + 1)\Delta \} \quad \text{for each } i = 2, \ldots, m_p. \quad (20)
\]

Note that with this mapping, for any \( t \in T \), there is a unique \( \epsilon \) which is the time period of exit from the \( i \)th arc on path \( p \) if the vehicle departed from the origin node in period \( t \); that is, the time period of entrance to the \( i+1 \)st arc on path \( p \). The actual time to traverse path \( p \) under traffic conditions \( (v,w) \) is calculated using the following nested function:

\[
D_{pe}(v,w) = D_{a_1}(v,w) + D_{a_2z'_{a_1}}(v,w) + \cdots + D_{a_{m_p}z'_{a_{m_p-1}}}(v,w). \quad (21)
\]

After de Palma et al. (1983), we introduce the nonnegative schedule delay cost function \( \phi_n(z'_{a_{m_p}} \Delta) \) which describes the penalty incurred if one departs the origin \( k \) and chooses a path \( p \).
expressed in (18) in period $t$ and arrives early or late at the destination $n$ via the last arc $a_{np}$ on path $p$. For simplicity, we assume that the generalized unit travel cost incurred by commuters departing their origin in period $t$ and choosing path $p$ is associated only with path travel time and schedule delay cost as follows:

$$c_{pj}(v, w) = \eta D_{pj}(v, w) + \phi_n(c_{a_{np}}^t, \Delta) \quad \forall p \in P_{kn}, \quad k \in K, \quad n \in N, \quad t \in T,$$

(22)

where $\eta$ is the value of travel time. We make the following definitions of minimum generalized unit travel costs:

$$\bar{\mu}_{kn} = \min \left\{ c_{pj}(v, w) : p \in P_{kn}, t \in T \right\} \quad \forall k \in K, \quad n \in N,$$

(23)

$$\bar{\psi}_{jt} = \min \left\{ c_{pj}(v, w) : p \in P_{jn} \right\} \quad \forall j \in M, \quad n \in N, \quad t \in T.$$

(24)

Including the flow propagation mechanism in (1)–(3), the arc traffic dynamics in (4)–(6), and the definitions of minimum generalized unit travel costs in (23) and (24), we formulate an arc-based discrete time dynamic network user equilibrium traffic assignment model with elastic demands as follows:

$$v_{ja}A \left[ \eta D_{at}(v, w) + \bar{\psi}_{inj} - \bar{\mu}_{kn} \right] = 0 \quad \forall a = (k, l) \in A(k), \quad k \in K, \quad l \in M, \quad n \in N, \quad t \in T,$$

(25)

$$\eta D_{at}(v, w) + \bar{\psi}_{inj} - \bar{\mu}_{kn} \geq 0 \quad \forall a = (k, l) \in A(k), \quad k \in K, \quad l \in M, \quad n \in N, \quad t \in T,$$

(26)

$$w_{ja}A \left[ \eta D_{at}(v, w) + \bar{\psi}_{inj} - \bar{\psi}_{jt} \right] = 0 \quad \forall a = (j, l) \in A(j), \quad j \in M, \quad l \in M, \quad n \in N, \quad t \in T,$$

(27)

$$\eta D_{at}(v, w) + \bar{\psi}_{inj} - \bar{\psi}_{jt} \geq 0 \quad \forall a = (j, l) \in A(j), \quad j \in M, \quad l \in M, \quad n \in N, \quad t \in T,$$

(28)

$$\sum_{a \in A(k)} v_{ja}A - Q_{jn}(\bar{\mu}) = 0 \quad \forall j \in M, \quad n \in N,$$

(29)

$$\sum_{a \in A(j)} w_{ja}A - \sum_{a \in B(j)} r_{an}(v, w)A = 0 \quad \forall j \in M, \quad n \in N, \quad t \in T,$$

(30)

$$v_{ja} \geq 0 \quad \forall a \in A, \quad n \in N, \quad t \in T,$$

(31)

$$w_{ja} \geq 0 \quad \forall a \in A, \quad n \in N, \quad t \in T,$$

(32)

$$\bar{\mu}_{kn} \geq 0 \quad \forall k \in K, \quad n \in N,$$

(33)

$$\bar{\psi}_{jt} \geq 0 \quad \forall j \in M, \quad n \in N, \quad t \in T,$$

(34)

$$\bar{\psi}_{jn} = \phi_n(t\Delta) \quad \forall n \in N, \quad t \in T.$$

(35)

The first four equations (25)–(28) model a dynamic generalization of Wardrop’s first principle requiring that the generalized unit travel costs be equilibrated for all departure time/route choices that are used during the entire interval of analysis. By definition, $\eta D_{at}(v, w) + \bar{\psi}_{inj}$ is the minimum cost of traveling from the origin $k$ to the destination $n$ via arc $a = (k, l) \in A(k)$, and $\bar{\mu}_{kn}$ is the minimum cost of traveling from the origin $k$ to the destination $n$ considering all departure time/route choices. If $\eta D_{at}(v, w) + \bar{\psi}_{inj} > \bar{\mu}_{kn}$, no vehicle would enter arc $a$ in period $t$ from the origin $k$ to reach the destination $n$. Since each vehicle is identified only by its destination node after departing from its origin node, it is important to ensure the equilibration of generalized unit travel
costs separately for the two different types of paths: one whose tail node is one of the origin nodes in the network and the other whose tail node is one of the en route junction nodes.

Conservation of travel demand between each origin–destination pair is ensured by (29), where \( A(k) \) is the set of arcs whose tail node is origin \( k \) and \( Q_{kn}(\tilde{\mu}) \) is the total travel demand between origin \( k \in K \) and destination \( n \in N \) during the time interval \( 0 \leq t \leq T \) determined as a function of the full vector of minimum generalized unit travel costs, \( \tilde{\mu} = [\tilde{\mu}_{kn} : k \in K, n \in N] \). Conservation of flows at each junction node is also ensured by (30), where \( A(j) \) is the set of arcs whose tail node is \( j \) and \( B(j) \) is the set of arcs whose head node is \( j \).

Eqs. (31)–(34) require that arc flows and minimum travel costs be nonnegative. Eq. (35) is definitional, implying that \( \psi_{ln} = \phi_n(z_a^j \Delta) \) when the arrival time period at the destination \( n \) is \( z_a^j \) and \( l = n \) which means that the head node of arc \( a \) is the destination node \( n \). In the rest of this paper, the dynamic network user equilibrium traffic assignment model with elastic demands formulated in (1)–(6) and (23)–(34) will be referred to as the model \( \mathcal{DNUCE} \).

We are now ready to define the notion of an equilibrium for simultaneous departure time choice and routing decisions when travel demands are elastic.

**Definition 1.** The time-varying arc flow and associated minimum cost pattern \((v^*, w^*, \mu^*, \psi^*)\) is a discrete time dynamic network user equilibrium if it satisfies conditions (1)–(6) and (23)–(34).

### 3. Nonlinear complementarity problem

In this section we show that the discrete time dynamic network user equilibrium model can be formulated as an equivalent nonlinear complementarity problem, which will be referred to as the problem \( \mathcal{NC} \). For simplification in the formulation, let \( y = (v, w, \mu, \psi) \). Also, let

\[
E_{ant}(y) = \eta D_A(v, w) + \psi_{ln} - \mu_{kn} \quad \forall a = (k, l) \in A(k), \quad k \in K, \quad l \in M, \quad n \in N, \quad t \in T, \quad (36)
\]

\[
F_{ant}(y) = \eta D_A(v, w) + \psi_{ln} - \psi_{jnt} \quad \forall a = (j, l) \in A(j), \quad j \in M, \quad l \in M, \quad n \in N, \quad t \in T, \quad (37)
\]

\[
G_{kn}(y) = \sum_{t \in T} \sum_{a \in A(k)} v_{ant} \Delta - Q_{kn}(\mu) \quad \forall k \in K, \quad n \in N, \quad (38)
\]

\[
H_{jnt}(y) = \sum_{a \in A(j)} w_{ant} \Delta - \sum_{a \in B(j)} r_{ant}(v, w) \Delta \quad \forall j \in M, \quad n \in N, \quad t \in T, \quad (39)
\]

\[
I_{nnt}(y) = \phi_n(t^A) - \psi_{nt} \quad \forall n \in N, \quad t \in T. \quad (40)
\]

We establish the following equivalent nonlinear complementarity formulation of the discrete time dynamic network user equilibrium problem with elastic demands:

**Theorem 1.** Suppose that \( D_A(v, w) \) is a positive function for all \( a \in A \) and \( t \in T \). Also suppose that \( r_{ant}(v, w) \) is a nonnegative function for all \( a \in A, \quad n \in N \) and \( t \in T \) and \( Q_{kn}(\mu) \) is a nonnegative function for all \( k \in K \) and \( n \in N \). The discrete time dynamic network user equilibrium model is equivalent to the following nonlinear complementarity problem: find \( y^* = (v^*, w^*, \mu^*, \psi^*) \) such that

\[
v_{ant}^* E_{ant}(y^*) = 0 \quad \forall a = (k, l) \in A(k), \quad k \in K, \quad l \in M, \quad n \in N, \quad t \in T, \quad (41)
\]
\[ E_{an}(y^*) > 0 \quad \forall a = (k, l) \in A(k), \quad k \in K, \quad l \in M, \quad n \in N, \quad t \in T, \tag{42} \]
\[ w_{an}^* A_{an}(y^*) = 0 \quad \forall a = (j, l) \in A(j), \quad j \in M, \quad l \in M, \quad n \in N, \quad t \in T, \tag{43} \]
\[ F_{an}(y^*) \geq 0 \quad \forall a = (j, l) \in A(j), \quad j \in M, \quad l \in M, \quad n \in N, \quad t \in T, \tag{44} \]
\[ \mu_{kn} G_{kn}(y^*) = 0 \quad \forall k \in K, \quad n \in N, \tag{45} \]
\[ G_{kn}(y^*) \geq 0 \quad \forall k \in K, \quad n \in N, \tag{46} \]
\[ \psi_{ja} H_{jnt}(y^*) = 0 \quad \forall j \in M, \quad n \in N, \quad t \in T, \tag{47} \]
\[ H_{jnt}(y^*) \geq 0 \quad \forall j \in M, \quad n \in N, \quad t \in T, \tag{48} \]
\[ \sum_{a \in B(n)} r_{an}(v^*, w^*) \Delta A_{an}(y^*) = 0 \quad \forall n \in N, \quad t \in T, \tag{49} \]
\[ y^* \geq 0. \tag{50} \]

**Proof.** The proof has two parts. The first part proves that any solution of the problem \( \mathcal{D} \mathcal{N} \mathcal{U} \mathcal{E} \) solves the problem \( \mathcal{N} \mathcal{E} \), and the second part proves that any solution of the problem \( \mathcal{N} \mathcal{E} \) solves the problem \( \mathcal{D} \mathcal{N} \mathcal{U} \mathcal{E} \). Let \( y^* = (v^*, w^*, \mu^*, \psi^*) \) be a discrete time dynamic network user equilibrium in the sense of Definition 1. Then \( y^* \) solves the problem \( \mathcal{N} \mathcal{E} \) because \( y^* \) satisfies \( G_{kn}(y^*) = 0 \) for all \( k \in K \) and \( n \in N \) and \( H_{jnt}(y^*) = 0 \) for all \( j \in M, n \in N \) and \( t \in T \) as well.

It suffices to show that any solution to the problem \( \mathcal{N} \mathcal{E} \) is a discrete time dynamic network user equilibrium in the sense of Definition 1. Let \( (v^*, w^*, \mu^*, \psi^*) \) be a solution to the problem \( \mathcal{N} \mathcal{E} \). But, suppose that (46) is not binding for some origin–destination pair as follows:

\[ \sum_{t \in T} \sum_{a \in A(k)} v_{an}^* A_{an} - Q_{kn}(\mu^*) > 0. \tag{51} \]

It follows from (45) that \( \mu_{kn}^* = 0 \) for that origin–destination pair. In addition, since \( Q_{kn}(\mu^*) \) is nonnegative, we know that

\[ \sum_{t \in T} \sum_{a \in A(k)} v_{an}^* A_{an} > Q_{kn}(\mu^*) \geq 0, \tag{52} \]

which implies that \( v_{an}^* > 0 \) for some departure time choice and routing decision. However, for this particular simultaneous choice, (41) implies that

\[ \eta D_{al}(v^*, w^*) + \psi_{lnt}^* = \mu_{kn}^*. \tag{53} \]

Since \( \mu_{kn}^* = 0 \), we know that \( \eta D_{al}(v^*, w^*) + \psi_{lnt}^* = 0 \) for this particular choice, which contradicts the assumption that \( D_{al}(v^*, w^*) \) and \( c_{pr}(v^*, w^*) \) are positive functions. Hence, (46) holds as an equality so that (29) holds.

Suppose that (48) is not binding for some junction–destination pair in some time period \( t \) as follows:

\[ \sum_{a \in A(j)} w_{an}^* A_{an} - \sum_{a \in B(j)} r_{an}(v^*, w^*) A_{an} > 0. \tag{54} \]

It follows from (47) that \( \psi_{jnt}^* = 0 \) for that junction–destination pair in that time period. In addition, since \( r_{an}(v^*, w^*) \) is nonnegative from (6), we know that
which implies that \( w^*_{a_{ij}} > 0 \) for at least one of the arcs whose tail node is \( j \) in time period \( t \). However, for this particular time, arc and destination tuple, (43) implies that
\[
\psi^*_j v^*_i = \psi^*_{i_{nt}}.
\]

Since \( \psi^*_{i_{nt}} = 0 \), we know that \( \eta D_{a_{ij}} (v^*, w^*) + \psi^*_{i_{nt}} = 0 \) for this particular time, arc and destination tuple, which contradicts the assumption that \( D_{a_{ij}} (v^*, w^*) \) and \( c_p (v^*, w^*) \) are positive functions. Hence, (48) holds as an equality so that (30) holds. It also follows from (49) that \( \psi^*_{i_{nt}} = \phi_n (t A) \) for all \( n \in N \) and \( t \in T \) if \( \sum_{a \in B(a)} r_{a_{nt}} (v^*, w^*) A > 0 \).

It still remains to show that \( (\mu^*, \psi^*) \) are identical to the minimum generalized unit travel costs defined in (23) and (24) under traffic conditions \((v^*, w^*)\). To this end, consider an arbitrary path expressed in (18) and show that \( c_p (v^*, w^*) = \min \{ c_p (v^*, w^*) : q \in P_k \} \) for any solution of the problem \( N^C \). First, it is known from (41) and (43) that if \( v^*_{a_{ij}} > 0 \) and \( w^*_{a_{inj}} > 0 \) for each \( i = 2, \ldots, m_p \) on a path for some departure time period \( t \in T \). It should be noted that if \( v^*_{a_{ij}} > 0 \) is replaced by \( w^*_{a_{ij}} > 0 \), \( c_p (v^*, w^*) = \psi^*_{i_{nt}} \). Hence, the theorem follows immediately.

It should be noted that the nonlinear complementarity problem \( N^C \) also includes the equality constraints (4) and (6) with initial conditions (5), which do not need complementarity conditions. It should also be noted that the variables \( x_{nt} \) do not appear explicitly in the problem \( N^C \) via recursive substitutions described in (17). Similarly, the equality constraint (6) can be used to substitute the variables \( r_{nt} (v, w) \) as appropriate sums of \( v_{nt} \) and \( w_{nt} \) in the problem \( N^C \). For simplicity in the rest of this paper, we will simply assume that these substitutions have been made.

### 4. Existence

In this section, the existence of a solution to the nonlinear complementarity problem \( N^C \) is established using Brouwer’s fixed-point theorem. Our approach is similar to one used in the work of Aashtiani and Magnanti (1981) in that the nonlinear complementarity problem \( N^C \) is converted into a Brouwer fixed-point problem. Brouwer’s fixed-point theorem, however, cannot be directly applied because the feasible solution set for the nonlinear complementarity problem \( N^C \)
is unbounded. Some notion of the boundness of the solution needs to be induced in order to establish the existence of at least one solution. Such a boundness can be accomplished by defining a bounded set $\Omega$ within the feasible solution set for the problem $\mathcal{NC}$ and showing that no point outside $\Omega$ or on the boundary of $\Omega$ is a candidate for solution. We will use the following definitions for simplicity when no confusion arises:

\[ D_{at} = D_{at}(v, w), \]  
\[ r_{ant} = r_{ant}(v, w). \]  

(60)

(61)

It is now possible to state and prove the following result:

**Theorem 2.** Suppose that $D_{at}(v, w)$ is a positive continuous function for all $a \in A$ and $t \in T$. Also suppose that $r_{ant}(v, w)$ is a nonnegative continuous function for all $a \in A$, $n \in N$ and $t \in T$ and $Q_{kn}(\mu)$ is a nonnegative continuous function that is bounded from above for all $k \in K$ and $n \in N$. Then the nonlinear complementarity problem $\mathcal{NC}$ has a solution.

**Proof.** First, we define the following notations:

\[ \mathcal{E} > \max \{ Q_{kn}(\mu) : \forall k \in K, n \in N \}, \]  
\[ \mathcal{F} > \max \left\{ \sum_{a \in B(j)} r_{ant}(v, w) A : \forall j \in M, n \in N, t \in T \right\}, \]  
\[ \mathcal{G} > \max \left\{ \eta D_{at} + \psi_{lnz_a} : \forall a = (k, l) \in A, l \in M, n \in N, t \in T \right\}. \]  

(62)

(63)

(64)

Note that $\mathcal{E} > 0$ exists because each $Q_{kn}(\mu)$ is bounded from above, $\mathcal{F}$ exists because $\mathcal{E}$ exists, and $\mathcal{G}$ exists because $\mathcal{E}$ and $\mathcal{F}$ exist. Next, we define $\Omega$ as a bounded set of feasible solutions to the problem $\mathcal{NC}$, satisfying

\[ 0 \leq v_{ant} A \leq \mathcal{E} \quad \forall a \in A, \ n \in N, \ t \in T, \]  
\[ 0 \leq w_{ant} A \leq \mathcal{F} \quad \forall a \in A, \ n \in N, \ t \in T, \]  
\[ 0 \leq \mu_{kn} \leq \mathcal{G} \quad \forall k \in K, \ n \in N, \]  
\[ 0 \leq \psi_{jnt} \leq \mathcal{G} \quad \forall j \in M, \ n \in N, \ t \in T. \]  

(65)

(66)

(67)

(68)

Since $\Omega$ is bounded and closed, it is compact and also convex (Rockafellar, 1970).

Based on the projection operator $[b]^+ = \max[0, b]$, we define the following fixed-point problem $\mathcal{F} \mathcal{P}^+$: Find $(v, w, \mu, \psi) \in \Omega$ such that

\[ v_{ant} A = \min \left\{ \mathcal{E}, \left[ v_{ant} A + \mu_{kn} - \eta D_{at} - \psi_{lnz_a} \right]^+ \right\} \quad \forall a = (k, l) \in A(k), \ k \in K, \ n \in N, \ t \in T, \]  
\[ w_{ant} A = \min \left\{ \mathcal{F}, \left[ w_{ant} A + \psi_{jnt} - \eta D_{at} - \psi_{lnz_a} \right]^+ \right\} \quad \forall a = (j, l) \in A(j), \ j \in M, \ n \in N, \ t \in T, \]  

(69)

(70)
\begin{equation}
\mu_{kn} = \min \left\{ \mathcal{H}, \left[ \mu_{kn} + Q_{kn}(\mu) - \sum_{t \in T} \sum_{a \in A(k)} v_{ant}A \right]^+ \right\} \quad \forall k \in K, \ n \in N,
\end{equation}

(71)

\begin{equation}
\psi_{jnt} = \min \left\{ \mathcal{H}, \left[ \psi_{jnt} + \sum_{a \in B(j)} r_{ant}A - \sum_{a \in A(j)} w_{ant}A \right]^+ \right\} \quad \forall j \in M, \ n \in N, \ t \in T.
\end{equation}

(72)

By Brouwer’s fixed-point theorem (see e.g., Todd, 1976), \( \mathcal{H} \mathcal{P} \mathcal{P}^+ \) has a fixed point \((v^*, w^*, \mu^*, \psi^*) \in \Omega \). We now wish to show that this fixed point solves the nonlinear complementarity problem \( \mathcal{N}^C \) by showing that

\begin{equation}
v_{ant}^*A = \left[ v_{ant}^*A + \mu_{kn}^* - \eta D_{at}^* - \psi_{lnta}^* \right]^+ \quad \forall a = (k, l) \in A(k), \ k \in K, \ n \in N, \ t \in T,
\end{equation}

(73)

\begin{equation}
w_{ant}^*A = \left[ w_{ant}^*A + \psi_{jnt}^* - \eta D_{at}^* - \psi_{lnta}^* \right]^+ \quad \forall a = (j, l) \in A(j), \ j \in M, \ n \in N, \ t \in T,
\end{equation}

(74)

\begin{equation}
\mu_{kn}^* = \left[ \mu_{kn}^* + Q_{kn}(\mu^*) - \sum_{t \in T} \sum_{a \in A(k)} v_{ant}^*A \right]^+ \quad \forall k \in K, \ n \in N,
\end{equation}

(75)

\begin{equation}
\psi_{jnt}^* = \left[ \psi_{jnt}^* + \sum_{a \in B(j)} v_{ant}^*A - \sum_{a \in A(j)} w_{ant}^*A \right]^+ \quad \forall j \in M, \ n \in N, \ t \in T.
\end{equation}

(76)

In other words, we need to show that a fixed point \((v^*, w^*, \mu^*, \psi^*) \) lies neither on the boundary of \( \Omega \) nor outside \( \Omega \).

We begin by establishing (73). To show that \( v_{ant}^*A < \mathcal{H} \) for all \( a \in A, n \in N, t \in T \), suppose some \( v_{ant}^* = \mathcal{H} \). Then \( \sum_{a \in A(k)} v_{ant}^*A > Q_{kn}(\mu^*) \) by the definition of \( \mathcal{H} \) in Eq. (62), which implies that \( \mu_{kn}^* + Q_{kn}(\mu^*) - \sum_{a \in A(k)} v_{ant}^*A < \mu_{kn}^* \) and \( \mu_{kn}^* = 0 \) from (71). By the positivity of \( \eta D_{at}^* + \psi_{lnta}^* \), we know that \( v_{ant}^*A + \mu_{kn}^* - \eta D_{at}^* - \psi_{lnta}^* < v_{ant}^*A \). Note that \( \psi_{lnta}^* = \phi_n(z_{lnt}^*) \) when \( l = n \) where \( \phi_n(z_{lnt}^*) \) is assumed to be nonnegative. Consequently, \( v_{ant}^* \) must be equal to zero in order that Eq. (69) holds, contradicting \( v_{ant}^*A = \mathcal{H} \). Thus \( v_{ant}^*A < \mathcal{H} \) hence (73) holds.

We proceed by establishing (74). To show that \( w_{ant}^*A < \mathcal{H} \) for all \( a \in A, n \in N, t \in T \), suppose some \( w_{ant}^*A = \mathcal{H} \). Then \( \sum_{a \in B(j)} w_{ant}^*A > \sum_{a \in B(j)} v_{ant}^*A \) by the definition of \( \mathcal{H} \) in Eq. (63), which implies that \( \psi_{jnt}^* + \sum_{a \in B(j)} v_{ant}^*A - \sum_{a \in A(j)} w_{ant}^*A < \psi_{jnt}^* \) and \( \psi_{jnt}^* = 0 \) from (72). By the positivity of \( \eta D_{at}^* + \psi_{lnta}^* \), we know that \( w_{ant}^*A + \psi_{jnt}^* - \eta D_{at}^* - \psi_{lnta}^* < w_{ant}^*A \). Consequently, \( w_{ant}^*A = \mathcal{H} \) must be equal to zero in order that Eq. (70) holds, contradicting \( w_{ant}^*A = \mathcal{H} \). Thus \( w_{ant}^*A < \mathcal{H} \) hence (74) holds.

We next establish (75). To show that \( \mu_{kn}^* < \mathcal{H} \) for all \( k \in K, \ n \in N \), suppose some \( \mu_{kn}^* = \mathcal{H} \). Then \( \mu_{kn}^* > \eta D_{at}^* + \psi_{lnta}^* \) by the definition of \( \mathcal{H} \) in Eq. (64), which implies that \( v_{ant}^*A + \mu_{kn}^* - \eta D_{at}^* - \psi_{lnta}^* > v_{ant}^*A \) and \( v_{ant}^*A = \mathcal{H} \) from (69). However, the definition of \( \mathcal{H} \) in (62) implies that \( \mu_{kn}^* + Q_{kn}(\mu^*) - \sum_{t \in T} \sum_{a \in A(k)} v_{ant}^*A < \mu_{kn}^* \). Consequently, \( \mu_{kn}^* \) must be equal to zero in order that Eq. (71) holds, contradicting \( \mu_{kn}^* = \mathcal{H} \). Thus \( \mu_{kn}^* < \mathcal{H} \) hence (75) holds.

We finally establish (76). To show that \( \psi_{jnt}^* < \mathcal{H} \) for all \( k \in K, \ n \in N, \) and \( t \in T \), suppose some \( \psi_{jnt}^* = \mathcal{H} \). Then \( \psi_{jnt}^* > \eta D_{at}^* + \psi_{lnta}^* \) by the definition of \( \mathcal{H} \) in Eq. (64), which implies that \( v_{ant}^*A + \psi_{jnt}^* - \eta D_{at}^* - \psi_{lnta}^* > v_{ant}^*A \) and \( v_{ant}^*A = \mathcal{H} \) from (70). However, the definition of \( \mathcal{H} \) in (63) implies that \( \psi_{jnt}^* + \sum_{a \in B(j)} v_{ant}^*A - \sum_{a \in A(j)} w_{ant}^*A < \psi_{jnt}^* \). Consequently, \( \psi_{jnt}^* \) must be equal to zero in

order that Eq. (72) holds, contradicting $\psi_{jnt} = \emptyset$. Thus $\psi_{jnt} < \emptyset$ hence (76) holds. This completes the proof. □

5. Uniqueness

As in the case of the static network user equilibrium problem where path flows need not be unique, destination-specific arc flows and volumes need not be unique in the dynamic case because two collections of destination-specific flows and volumes may correspond to the same total arc flow and volume, respectively. Therefore, a destination-specific solution to the nonlinear complementarity problem $NCE$ need not be unique. In this section, we show that if a destination-specific solution to the problem $NCE$ is aggregated over all destinations, it is a unique discrete time dynamic network user equilibrium of Definition 1. Such a unique equilibrium solution is represented by the total arc inflow vector $\bar{u} = [u_{at} : a \in A, t \in T]$, the total arc outflow vector $\bar{r} = [r_{at} : a \in A, t \in T]$, the total arc traffic volume vector $\bar{x} = [x_{at} : a \in A, t \in T]$, and the associated minimum generalized unit travel cost vectors $\mu$ and $\psi$. Note that $u_{at}$ is determined by using the relationships in (8) and (11). Our approach to show the uniqueness is similar to one used in the work of Aashtiani and Magnanti (1981) in that strict monotonicity conditions are imposed on both the arc travel time functions and the origin–destination travel demand functions.

Suppose that $(x, u)$ and $(x', u')$ are two distinct dynamic equilibrium volume and inflow patterns that are aggregated solutions to the nonlinear complementarity problem $NCE$. We introduce the following definition of regularity:

**Definition 2.** The functions $C_a(x_{at}, u_{at})$ are said to be strictly monotone increasing in $x_{at}$ and $u_{at}$ if, for every pair $(x_{at}, u_{at})$, $(x'_{at}, u'_{at})$ with $(x_{at}, u_{at}) \neq (x'_{at}, u'_{at})$, for all $a \in A$ and $t \in T$,

$$
(x_{at} - x'_{at})[C_a(x_{at}, u_{at}) - C_a(x'_{at}, u'_{at})] + (u_{at}A - u'_{at}A)[C_a(x_{at}, u_{at}) - C_a(x'_{at}, u'_{at})] > 0.
$$

(77)

Note that this is the same as requiring that $C_a(x_{at}, u_{at})$ be a strictly monotonic function of $x_{at} + u_{at}A$. However, because $x_{at}$ depends only on $u_{at}A$ in previous periods, either of the two terms on the left-hand side of the inequality could be zero or negative. We require an additional smoothness condition on the equilibrium problem as a whole. This condition generally holds because of the form of the schedule delay cost function imposed on each destination. It requires that for any two distinct equilibrium flow patterns, $(u, x)$ and $(u', x')$

$$
\sum_{t \in T} \sum_{a \in A} (u_{at}A - u'_{at}A)[C_a(x_{at}, u_{at}) - C_a(x'_{at}, u'_{at})] > 0.
$$

(78)

For this smoothness condition to hold it requires that, over most time periods, if $u_{at}A > u'_{at}A$ then $x_{at} + u_{at}A > x'_{at} + u'_{at}A$. This will be the case except possibly when the input flows exhibit large changes from increasing to decreasing (or vice versa). Because of the nature of the schedule delay cost function, equilibrium flows to a destination on an arc will increase to a peak and then subside. Superposition of all destination flows on an arc may cause several peaks, but if the time periods are small enough, any reversals of sign of volumes differences and input differences
between two equilibria will only occur for a small number of time periods, and so the sum in (78) will be positive.

We also introduce the following definition of regularity:

**Definition 3.** The functions \( Q_{kn}(\mu) \) are said to be strictly monotone decreasing in \( \mu \) if, for every pair \( \mu, \mu' \) with \( \mu \neq \mu' \), for all \( k \in K \) and \( n \in N \),

\[
(\mu_{kn} - \mu'_{kn})(Q_{kn}(\mu) - Q_{kn}(\mu')) < 0. \tag{79}
\]

It is now possible to state and prove the following uniqueness result:

**Theorem 3.** Suppose that \( C_a(x_{at}, u_{at}) \) is strictly monotone increasing in \( x_{at} \) and \( u_{at} \) for all \( a \in A \) and \( t \in T \) and \( Q_{kn}(\mu) \) is strictly monotone decreasing in \( \mu \) for all \( k \in K \) and \( n \in N \). Also suppose that any dynamic equilibrium flow pattern satisfies the condition of smoothness. Then the nonlinear complementarity problem \( N' \mathcal{C} \) has a unique aggregated solution \((\bar{u}, \bar{r}, \bar{x}, \mu, \psi)\).

**Proof.** We know from (41), (43), (45), (47) and (49) that

\[
\sum_{t \in T} \sum_{a=(k,l) \in A(k)} \sum_{k \in K} \sum_{n \in N} v_{am} A \left[ \eta D_{at}(v, w) + \psi_{lna} - \mu_{kn} \right] \\
+ \sum_{t \in T} \sum_{a=(j,l) \in A(j)} \sum_{j \in M} \sum_{n \in N} w_{am} A \left[ \eta D_{at}(v, w) + \psi_{lna} - \psi_{jnt} \right] \\
+ \sum_{k \in K} \sum_{n \in N} \mu_{kn} \left[ \sum_{t \in T} \sum_{a \in A(k)} v_{am} A - Q_{kn}(\mu) \right] + \sum_{t \in T} \sum_{j \in M} \sum_{n \in N} \psi_{jnt} \left[ \sum_{a \in A(j)} w_{am} A - \sum_{a \in B(j)} r_{am}(v, w) A \right] \\
+ \sum_{t \in T} \sum_{n \in N} \sum_{a \in B(n)} r_{am}(v, w) A [\phi_n(tA) - \psi_{mnt}] = 0. \tag{80}
\]

Eliminating identical terms in (80) and using the relationships in (8)–(10) yields

\[
\sum_{t \in T} \sum_{a \in A} \eta u_{at} AD_{at}(v, w) + \sum_{t \in T} \sum_{a=(k,l) \in A(k)} \sum_{k \in K} \sum_{n \in N} u_{am} A \psi_{lna} - \sum_{k \in K} \sum_{n \in N} \mu_{kn} Q_{kn}(\mu) \\
- \sum_{t \in T} \sum_{j \in M} \sum_{n \in N} \sum_{a \in B(j)} r_{am}(v, w) A \psi_{jnt} - \sum_{t \in T} \sum_{n \in N} \sum_{a \in B(n)} r_{am}(v, w) A \psi_{mnt} \\
+ \sum_{t \in T} \sum_{n \in N} \sum_{a \in B(n)} r_{am}(v, w) A \phi_a(tA) = 0. \tag{81}
\]

Suppose that \((v, w, \mu, \psi)\) and \((v', w', \mu', \psi')\) are two distinct solutions to the nonlinear complementarity problem \( N' \mathcal{C} \). As noted previously, \( D_{at}(v, w) \) is set equal to \( C_a(x_{at}, u_{at}) \) when the network is loaded according to \((v, w)\) following the prespecified traffic dynamics and flow propagation mechanism. Using (8), \( r_{am}(v, w) \) is expressed as \( r_{am}(u) \). The complementarity conditions (41)–(48) and nonnegativity condition (50) imply that
\[
\sum_{t \in T} \sum_{a \in A} \eta(u_{at} - u'_{at}) \left[ C_a(x_{at}, u_{at}) - C_a(x'_{at}, u'_{at}) \right] \Delta \\
+ \sum_{t \in T} \sum_{a=(k,l) \in A(k)} \sum_{k \in K} \sum_{n \in N} (u_{ant} - u'_{ant}) \left( \psi_{lnz_a} - \psi'_{lnz_a} \right) \Delta - \sum_{k \in K} \sum_{n \in N} (\mu_{kn} - \mu'_{kn}) \left[ Q_{kn}(\mu) - Q_{kn}(\mu') \right] \\
- \sum_{t \in T} \sum_{j \in M} \sum_{n \in N} \sum_{a \in B(j)} [r_{ant}(u) - r_{ant}(u')] \left( \psi_{jnt} - \psi'_{jnt} \right) \Delta \\
- \sum_{t \in T} \sum_{n \in N} \sum_{a \in B(n)} [r_{ant}(u) - r_{ant}(u')] \left( \psi_{mnt} - \psi'_{mnt} \right) \Delta \leq 0.
\]

(82)

We know from (2) that a single exit time can be mapped into consecutive entrance time periods because arc delays can be decreasing. We also know from the equilibrium conditions that if \( u_{aaa} > 0 \), then \( \eta C_{a,s}(x_{aa}, u_{aa}) + \psi_{lna} = \mu_{ka} \) for all \( a = (k, l) \in A(k), k \in K, n \in N, \) and \( \pi \in \theta_a^t \) and if \( u_{aaa} > 0 \), then \( \eta C_{a,a}(x_{aa}, u_{aa}) + \psi_{lna} = \mu_{kn} \) for all \( a = (k, l) \in A(k), k \in K, n \in N, \) and \( \pi \in \theta_a^t \). Thus, by using the traffic dynamics and flow propagation mechanism described in (1)-(6), we know that the second term in (82) can be rewritten as

\[
\sum_{t \in T} \sum_{a=(k,l) \in A(k)} \sum_{k \in K} \sum_{n \in N} (u_{ant} - u'_{ant}) \left( \psi_{lnz_a} - \psi'_{lnz_a} \right) \Delta \\
= \sum_{t \in T} \sum_{a=(k,l) \in A(k)} \sum_{k \in K} \sum_{n \in N} \left[ \sum_{\pi \in \theta_a^t} (1 - \rho_{ant}) u_{ann} + \sum_{a \in \theta_a^t} \rho_{a,a,t-1} u_{ano} - \sum_{\pi' \in \theta_a^t} (1 - \rho'_{a,a'} u_{ann'}) \right] \\
- \sum_{\pi' \in \theta_a^t} \rho'_{a,a',t-1} u_{ano'} \left( \psi_{lna} - \psi'_{lna} \right) \Delta.
\]

(83)

For simplicity, and without loss of generality, we assume that each arc belonging to either the set \( B(j) \) or the set \( B(n) \) has \( k \) as the tail node. Substituting (6) to the last two terms in (82) makes clear that they are identical to (83). It follows immediately that

\[
\sum_{t \in T} \sum_{a \in A} \eta(u_{at} - u'_{at}) \left[ C_a(x_{at}, u_{at}) - C_a(x'_{at}, u'_{at}) \right] \Delta - \sum_{k \in K} \sum_{n \in N} (\mu_{kn} - \mu'_{kn}) \left[ Q_{kn}(\mu) - Q_{kn}(\mu') \right] \leq 0.
\]

(84)

But (84) contradicts (78) and (79). Thus it implies that \( \bar{u} = \bar{u}' \), and \( \mu = \mu' \). Hence, the total arc inflow vector \( \bar{u} \) and the minimum generalized unit travel cost vector \( \mu \) are unique. The unique \( \bar{u} \) yields a unique total arc outflow vector \( \bar{r} \) from the definitions in (6) and (12), a unique total arc traffic volume vector \( \bar{x} \) from (4), (5), (13) and (17), and a unique \( \psi \) from the definition in (24). This completes the proof. \( \square \)

6. Iterative algorithm

In this section we present an iterative algorithm for solving an arc-based nonlinear complementarity formulation of the discrete time dynamic network user equilibrium problem. The proposed algorithm is heuristic in that convergence is not established by certain regularity
conditions. The main reason for using a heuristic algorithm to solve the problem is that the mapping \( D_{at}(v, w) \) is not a function that can be expressed in closed form, but rather is a mapping that requires the dynamic network loading in accordance with \((v, w)\). The heuristic algorithm is an effort to avoid having to evaluate the mapping too many times as is required by nonlinear complementarity algorithms. For simplicity, we assume that the travel demand function is separable and invertible, that is, \( S_{kn} = Q_{kn}(\mu_{kn}) \) and \( \mu_{kn} = Q_{kn}^{-1}(S_{kn}) \) for all \( k \in K \) and \( n \in N \).

Let \((v', w', \mu', \psi')\) be a feasible solution known at the \(i\)th iteration of the algorithm. Also let \( S_{kn}^{\text{max}} \) be an upper bound on the total travel demand between origin \( k \) and destination \( n \). To find an auxiliary flow pattern \((\tilde{v}, \tilde{w})\), the following linear program can be formulated:

\[
\begin{align*}
\min_{v, w} & \sum_{t \in T} \sum_{a \in A(k)} \sum_{k \in K} \sum_{n \in N} \tilde{v}_{ant}^t d_{at}(v', w') + \psi_{lnz}^t - Q_{kn}^{-1}(S_{kn}) \\
+ & \sum_{t \in T} \sum_{a \in A(j)} \sum_{j \in M} \sum_{n \in N} \tilde{w}_{ant}^t d_{at}(v', w') + \psi_{lnz}^t - \psi_{jnt}^t,
\end{align*}
\]

subject to

\[
\begin{align*}
\sum_{t \in T} \sum_{a \in A(k)} \tilde{v}_{ant}^t \Delta & \leq S_{kn}^{\text{max}} \quad \forall k \in K, \ n \in N, \quad (86) \\
\sum_{a \in A(j)} \tilde{w}_{ant}^t \Delta & \leq \sum_{a \in B(j)} r_{ant}(v', w') \Delta \quad \forall j \in M, \ n \in N, \ t \in T, \quad (87) \\
\tilde{v}_{ant}^t & \geq 0 \quad \forall a \in A(k), \ k \in K, \ n \in N, \ t \in T, \quad (88) \\
\tilde{w}_{ant}^t & \geq 0 \quad \forall a \in A(j), \ j \in M, \ n \in N, \ t \in T. \quad (89)
\end{align*}
\]

Note that each term in brackets in the objective function (85) is constant with respect to \( \tilde{v} \) and \( \tilde{w} \). This linear program calls for minimization of the total generalized travel cost over a whole network with fixed (not flow-dependent) unit travel costs. The total cost can be minimized using the all-or-nothing assignment procedure that assigns all trips to a route-departure time pair with the minimum cost for each origin–destination pair. Note that a route is distinguished only by its first arc to be traversed from an origin node or an intersection node and all downstream arcs after the first arc need not be specified to form a route. Therefore, an auxiliary flow pattern \((\tilde{v}, \tilde{w})\) can be determined simply by the following minimization principle:

\[
\begin{align*}
\text{If } & \eta D_{at}(v', w') + \psi_{lnz}^t \leq Q_{kn}^{-1}(S_{kn}'), \text{ set } \tilde{v}_{ant}^t \Delta = S_{kn}^{\text{max}}. \quad (90) \\
\text{If } & \eta D_{at}(v', w') + \psi_{lnz}^t > Q_{kn}^{-1}(S_{kn}'), \text{ set } \tilde{v}_{ant}^t \Delta = 0 \quad \forall a \in A(k), \ k \in K, \ n \in N, \ t \in T. \quad (91)
\end{align*}
\]

and

\[
\begin{align*}
\text{If } & \eta D_{at}(v', w') + \psi_{lnz}^t = \psi_{jnt}, \text{ set } \tilde{w}_{ant}^t \Delta = \sum_{a \in B(j)} r_{ant}(v', w') \Delta. \quad (92) \\
\text{If } & \eta D_{at}(v', w') + \psi_{lnz}^t > \psi_{jnt}, \text{ set } \tilde{w}_{ant}^t \Delta = 0 \quad \forall a \in A(j), \ j \in M, \ n \in N, \ t \in T. \quad (93)
\end{align*}
\]

Once the auxiliary flow pattern is known, the next solution can be generated by the following convex combination steps:
\[ v^{i+1}_a = v^i_a + \Theta^i_1 \left( \hat{v}^i_a - v^i_a \right) \quad \forall a \in A, \ n \in N, \ t \in T, \] (94)

\[ w^{i+1}_a = w^i_a + \Theta^i_2 \left( \hat{w}^i_a - w^i_a \right) \quad \forall a \in A, \ n \in N, \ t \in T, \] (95)

where \( 0 \leq \Theta^i_1 \leq 1 \) and \( 0 \leq \Theta^i_2 \leq 1 \) are the predetermined step sizes. Note that \( S^i_{ln} \) can be determined by \( \sum_{t \in T} \sum_{a \in A(k)} v^{i+1}_a \). Also note that once \( v^{i+1} \) and \( w^{i+1} \) are updated, \( \tau^{i+1} \) can be determined by the definition in (6) and \( x^{i+1} \) can be determined by solving the difference equations (4) and (5).

Details of the iterative algorithm are described as follows:

Step 1. Initialization: Choose the initial flow pattern \((v', w')\) and perform the dynamic network loading as described in (1)–(17). Compute the corresponding minimum cost pattern \((\mu', \psi')\). Set the iteration index \( i = 1 \).

Step 2. Descent direction finding: Find an auxiliary flow pattern \((\bar{v}', \bar{w}')\) that solves the linear program (85)–(89).

Step 3. Updating and convergence test: Find \( v^{i+1} \) and \( w^{i+1} \) by using (94) and (95). Find \( x^{i+1} \) by solving the difference equations (4)–(6) forward in time. If \( u' \approx u^{i+1} \), stop. Otherwise, set \( i = i + 1 \) and go to Step 2. To set up the linear program (85)–(89), compute the data items \( D_{at}(v^{i+1}, w^{i+1}), \psi_{lna}^{i+1} \), and \( Q_{ln}^{-1}(S^i_{ln}) \) using \((v^{i+1}, w^{i+1}, x^{i+1})\).

**7. Numerical results**

In this section the iterative algorithm is implemented on a test network that consists of 4 arcs, 2 origins, and 1 destination as follows: The network has four alternative paths between node 1 and node 3 that have some overlapping arcs. It is the same network used in Wie et al. (1995a,b). Time-varying flows and costs will be used to demonstrate that the proposed algorithm produces an approximate solution consistent with the discrete time dynamic network user equilibrium of Definition 1. The two-hour interval of the analysis is divided into 120 time periods of one minute length.

We know from (17) that \( x_{at} \) can be expressed as a sum of \( v_{at}, w_{at}, \) and \( r_{at}(v, w) \) from all earlier time periods via recursive substitutions. We also know that \( u_{at} \) is the sum of \( v_{at} \) and \( w_{at} \) as defined in (8). By assuming that these substitutions have already been made, we can then express the arc travel time function for each \( a = 1, \ldots, 4 \) and \( t = 0, \ldots, 120 \) in the following simple form:

\[ C_a(x_{at}, u_{at}) = \gamma_a + \zeta_a x_{at} + \lambda_a(x_{at})^2 + v_{at} u_{at} A + \xi_a(u_{at} A)^2, \] (96)
where \( \gamma_a \) is the free-flow travel time on arc \( a \), \( \Delta \) is the length of each time period that is assumed to be one-minute long, and \( \xi_a, \lambda_a, v_a, \) and \( \phi_a \) are the parameters. In the numerical example, we assume that \( D_{at}(v, w) \) is set equal to \( Ca(x_{at}, u_{at}) \). The values of these parameters are given in Table 1.

Let \([\bar{t} - \delta, \bar{t} + \delta]\) denote the desired time interval for arrival at each destination in the network where \( \bar{t} \) is the center of the interval and \( \delta \) measures the flexibility of arrival time. The unit schedule delay cost of arriving at the destination \( n \) in period \( t \) is predetermined as follows:

\[
\phi_n(t\Delta) = \begin{cases} 
\alpha(\bar{t} - \delta - t\Delta) & \text{if } \bar{t} - \delta > t\Delta, \\
\beta(t\Delta - \bar{t} - \delta) & \text{if } \bar{t} + \delta < t\Delta, \\
0 & \text{otherwise},
\end{cases}
\]

(97)

<table>
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<th>Arc number</th>
<th>( \gamma_a )</th>
<th>( \xi_a )</th>
<th>( \lambda_a )</th>
<th>( v_a )</th>
<th>( \phi_a )</th>
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</tr>
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</tr>
<tr>
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<td>0.2</td>
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</tr>
<tr>
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<td>0.1</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Fig. 1. Convergence of dynamic user equilibrium criterion.
Fig. 2. Total arc inflows.

Fig. 3. Total arc traffic volumes.
Fig. 4. Departure arc inflows and minimum costs.

Fig. 5. Departure arc inflows and minimum costs.
where $\alpha$ and $\beta$ are early and late arrival penalty coefficients, respectively. The values of coefficients in the unit schedule delay cost function in (97) are assumed that $\alpha = 0.5$, $\beta = 2.4$, $\delta = 0$, and $t = 80$. For simplicity, the value of travel time, $\eta$ in Eq. (22), is assumed to be unity. This assumption means that travel costs are equivalent to travel times with units of minutes. For simplicity, the travel demand functions are assumed to be separable and linear as follows:

$$Q_{13}(\mu_{13}) = 3000 - 60\mu_{13},$$

$$Q_{23}(\mu_{23}) = 2000 - 60\mu_{23}.$$  

(98)

(99)

Let $Z_{\text{DUE}}(v^i, w^i, \mu^i, \psi^i)$ denote the dynamic equilibrium criterion function that measures how closely a solution of the $i$th iteration satisfies the dynamic network user equilibrium conditions of Definition 1. This equilibrium criterion is defined as

$$Z_{\text{DUE}}(v^i, w^i, \mu^i, \psi^i) = \sum_{i \in I} \sum_{a \in A(k)} \sum_{k \in K} \sum_{n \in N} v^i_{ant} A \left[ \eta D_{at}(v^i, w^i) + \psi^i_{\text{int}} - \mu^i_{kn} \right]$$

$$+ \sum_{i \in I} \sum_{a \in A(j)} \sum_{j \in M} \sum_{n \in N} w^i_{ant} A \left[ \eta D_{at}(v^i, w^i) + \psi^i_{\text{int}} - \psi^i_{\text{fnt}} \right].$$

(100)

Fig. 1 shows that the value of the equilibrium criterion is monotonically decreasing as the iteration of the algorithm proceeds. Note that its value is equal to zero at an exact equilibrium solution. Figs. 2 and 3 present time trajectories of total arc inflows and total arc traffic volumes on arcs 1–4, respectively. Figs. 4 and 5 compare time trajectories of departure arc inflows and minimum unit travel costs. The equilibration of generalized unit travel costs is illustrated in Fig. 4 for vehicles departing from node 1 and traveling to node 3 and in Fig. 5 for vehicles departing from node 2 and traveling to node 3, respectively. It is ensured that there are positive departure inflows on arcs 1–4 only if the generalized unit travel costs from the origin node to the destination node via their respective arcs are equal to the minimum travel cost.

8. Conclusion

We have shown that a discrete time dynamic network user equilibrium problem with elastic demands can be formulated as an arc-based nonlinear complementarity problem. We have also shown that a dynamic equilibrium solution exists by using Brouwer’s fixed point theorem and its aggregated solution is unique by imposing strict monotonicity conditions on the arc travel cost and demand functions along with a smoothness condition on the equilibria. A heuristic iterative algorithm has been proposed to solve the problem without performing route enumeration and without storing path-specific flow and cost information. A number of important issues still remain to be resolved. First, it would be useful to establish a result that shows rigorously that the smoothness condition on the equilibria follows from certain conditions on the form of the schedule delay functions and the arc delay functions. Second, it is necessary to establish the conditions that guarantee the convergence of the algorithm. Third, a scheme for accelerating the convergence of the linearly convergent algorithm needs to be developed for savings in computation time.
References


