EXTENDING A TRAIN PATHING MODEL FROM ONE-WAY TO TWO-WAY TRACK

MALACHY CAREY
Faculty of Business and Management, University of Ulster, Northern Ireland, BT37 0QB, United Kingdom
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Abstract—A train pathing model and algorithms for a rail network was presented and tested in a previous paper. There, it was assumed that each rail line has two or more tracks and each was dedicated to traffic in one direction as is usual in Europe. Here, we show how to adapt and extend that model and algorithms so as to handle trains on single line two-way track, as is usual in North America. Though introducing two-way track makes the formulation appear more complex, I show that somewhat surprisingly, (a) it does not increase the number of constraints or variables, including O-1 variables, and (b) the model with two-way track should generally be easier to solve.

1. INTRODUCTION

Carey and Lockwood (1992) introduced and tested a basic model and algorithms for train pathing, and Carey (1994) extended and tested this work to allow for choice of lines, station platforms, routes, etc. In all of this it is assumed that each track is normally dedicated to trains in one direction—neighbouring stations are directly connected by at least double track rail having one or more lines in each direction. This allows track to be more heavily utilized, as it allows several trains on the same stretch of track at the same time, so long as adequate headways between them are maintained. This is the usual situation in the high density timetabled passenger traffic which is typical in Europe including Britain.

However, in North America and in some isolated parts of Europe and elsewhere, it is normal to have single lines with two-way traffic, with sidings at intervals to allow trains to meet and pass. In such two-way working, trains on a link from A to B have to exit at B before trains from B to A can enter. This greatly reduces the line capacity and is usually suitable only where trains are infrequent.

Though the models and algorithms in Carey make few explicit references to two-way links, these can be introduced. We do this by applying headway constraints analogous to those already set out there. In Section 2 we reintroduce some necessary notation, constraints, etc., from Carey, and use this to introduce two-way links into their general train pathing model $P^M$. This model is solved by reducing it to solving a sequence of subproblems $P^M$. In Section 3, we show how to handle two-way track in these subproblems $P^M$. In conclusion, we note that if some (or all) links are two-way rather than one-way the number of variables and constraints is not increased, the same algorithms as before apply, and the computational difficulty of the problem is likely to be reduced.

Recall that train pathing is the problem of assigning trains and train times, for a set of rail links, stations stops, etc., so as to meet a system of constraints on headways, trip times, dwell times, etc., while meeting travel demands and minimizing delays or costs.

2. A MODEL WITH MULTIPLE LINES AND PLATFORMS

A train pathing model $P^M$ for a rail network is set out in Section 3 of Carey. To extend this to "two-way track" we introduce two-way track headway constraints, stating these as special cases of headway constraints already defined in Section 3.2 of Carey. For this, we need the following notation from Section 3.1 of Carey.

1Carey refers to Carey (1994) throughout, unless stated otherwise.
Indices: Let \( u \) and \( v \) denote trains, \( s \) denote a link joining two nodes (timing points). Let \( L \) be the set of all links, and \( L_v \) be the set of links which train \( v \) is allowed to use. Let \( L^p \) be the set of (one-way) links on each of which only one train at a time is allowed. Let \( L^+ \) be the set of links on which two-way traffic is allowed. Let \( V \) be the set of all trains and \( V_s \) be the set of trains which are allowed to use link \( s \).

Variables: Let \( a_{us} \) be the time at which train \( v \) arrives on (enters) link \( s \), and \( d_{us} \) be the time at which it departs from link \( s \). Let \( x_{uv} \) be a 0–1 integer variable which takes the value 1 if train \( u \) immediately precedes train \( v \) on link \( s \), otherwise takes the value 0.

Parameters/constants: Let \( K_{us}^a \) be the minimum time interval (headway) required between the departure of train \( v \) and the departure of the preceding train \( u \) on link \( s \). Similarly, let \( K_{us}^d \) be the headway required between the arrival of train \( v \) and the arrival of the preceding train \( u \) on link \( s \). If link \( s \) can hold only one train at a time, let \( K_{us}^{da} \) be the minimum headway required between the departure time \( d_{us} \) of train \( u \) from the link and the arrival time \( a_{vs} \) of the next train \( v \) on the link. Let \( \delta_{uv} = 1 \) if train \( u \) is permitted to (immediately) precede train \( v \) on link \( s \), otherwise \( \delta_{uv} = 0 \). Let \( M \) be an arbitrarily large constant.

We now introduce headway constraints for one-way track (from Section 3.2 of Carey), which we will use to introduce headway constraints for two-way track.

2.1. One-way track constraints

For reasons of signalling, safety, etc., a minimum time interval (headway) is required between trains on each link, and because trains can travel at very different speeds, it is necessary to enforce these headways on trains on entering and exiting from each link. Thus:

Headway on entering (arriving on) a link. To ensure at least the minimum headway \( K_{us}^a \) between the arrival times \( a_{us} \) and \( a_{vs} \) of trains \( u \) and \( v \) on link \( s \), write

\[
a_{us} + K_{us}^a \leq a_{vs} + (1 - x_{uv})M \quad \text{for all } (u,v,s) \in V_s \delta_{uv} = 1
\]  

(3.4a)

The \((1 - x_{uv})M\) term ensures that the constraint is enforced if and only if train \( u \) immediately precedes train \( v \) on link \( s \).

Headway on exiting (departing) from a link. These are the same as the above constraints except that here all arrival times \( a's \) are replaced by departure times \( d's \). Thus

\[
d_{us} + K_{us}^d \leq d_{vs} + (1 - x_{uv})M \quad \text{for all } (u,v,s) \in V_s \delta_{uv} = 1
\]  

(3.4b)

One train at a time on some links, e.g., at platforms, some short links and "block signalled" sections. Most platforms at stations allow only one train at a time: the next train \( v \) is not allowed to enter until the preceding train \( u \) has exited, and a minimum headway \( K_{us}^{da} \) has elapsed. A similar restriction is required for so called block signalled sections of track. This can be written as,

\[
d_{us} + K_{us}^{da} \leq a_{vs} + (1 - x_{uv})M \quad \text{for all } (s \in L^p, u, v) \delta_{uv} = 1
\]  

(3.5)

where \( L^p \) denotes the set of links on each of which only one train at a time is allowed. The \((1 - x_{uv})M\) term ensures that these constraints are enforced if and only if train \( u \) immediately precedes \( v \) on link \( s \).

2.2. Two-way track constraints

Two-way links can now be introduced as follows.

(a) For links \( s \in L^+ \), on which two-way working is allowed, divide the set of trains \( V_s \)

\[
To facilitate comparison with Carey (1994), we use his equation numbers for all equations taken from there [i.e., equations (3.4a)-(3.5), (4.4a)-(4.5c)], and we number the remaining equations here [(3.10a)-(3.10b), (4.10a)-(4.10i)] so as to follow on from the equation numbers there.
into two sets: $V_+ = \{v \in V \mid v \text{ traverses link } s \text{ in the } + \text{ direction}\}$, and $V_- = \{v \in V \mid v \text{ traverses link } s \text{ in the } - \text{ direction}\}$.

(b) For all pairs of trains travelling in the same direction on links $s$, i.e., for all trains $(u, v)$ such that $[(u \in V_+, v \in V_+)$ or $(u \in V-, v \in V_-)$], the headway constraints are as for one-way links.

(c) Introduce a new set of headway constraints for all pairs of trains travelling in opposite directions on link $s$, i.e., for all trains $(u, v)$ such that $[(u \in V_-, v \in V_+)$ or $(v \in V_+, u \in V_-)$]. These constraints state that train $u$ can not enter link $s$ until train $v$ has exited from the link and vice versa. Thus these constraints are of the same form as the “One train at a time . . . ” constraints (3.5) above, except that here the constraints apply only if $s \in L_{+-}$ and $[[(u \in V_-, v \in V_+)$ or $(v \in V_+, u \in V_-)$].

The remaining constraints and cost terms in the model $P^M$ are unchanged.

2.3. Optimization model $P^M$

The train planning and pathing problem $P^M$ is now as already set out in Section 3.4 of Carey, plus the following two-way track headway constraints:

As in (3.4) above for all $s \in L_{+-}$ and
\[
(u,v)[[(u \in V_+, v \in V_+)$ or $(u \in V-, v \in V_-)]
\]
\[\quad (3.10a)\]
As in (3.5) above for all $s \in L_{+-}$ and
\[
(u,v)[[(u \in V_+, v \in V_-)$ or $(u \in V-, v \in V_+)]
\]
\[\quad (3.10b)\]

3. Reducing problem $P^M$ to problems $P^V$

In Carey, program $P^M$ is not solved directly. The algorithms used for solving program $P^M$, are based on solving a series of subproblems $P^V$, for each train $v$ (see Carey & Lockwood, 1992; Carey, 1994). Each program $P^V$ is obtained from program $P^M$ as follows: hold fixed the sequence order (but not the times) of all trains, except the “current” train $v$, on all links. We adopt the same definition here, and introduce the following notation from Section 5.1 of Carey. Let,

$U_ = \{u \in U \mid u \neq v\}$ which are already assigned to link $s$ (by previous programs $P^V$).

$u^+ = \{u \in U \mid u \text{ is the train immediately after train } u \text{ in the currently given ordering } U_\}$ of trains on link $s$. (Strictly this should be $u^+$ but using $U^+$ will not cause any confusion).

Let $v_+$ denote the final train on link $s$, in the ordered set $U_$. Introduce variables.

\[
w_{uv} = \begin{cases} 1 & \text{if train } u \text{ precedes train } v \text{ on link } s \text{ (i.e., train } v \text{ is slotted in between trains } u \\ \text{and } u^+ \{	ext{on link } s) \\ 0 & \text{otherwise.} \end{cases}
\]

Again as in Section 2, the easiest way to introduce two-way track into program $P^M$ is to set out the headway constraints for one-way track (from Section 5.2 of Carey) and use these to introduce headway constraints for two-way track. We therefore briefly restate the former constraint here.

3.1. One-way track constraints

\textbf{Headway on entering a link.} For program $P^M$, (3.4a) reduces to two sets of constraints, one set for the current train $v$ and one for the remaining trains $u \neq v$. For train $v$,

\[
a_{uv} + K_{uv}^s \leq a_{uv} + (1 - w_{uv})M \quad \text{for all } u \in U_+, s \in L, \delta_{uv} = 1 \quad (4.4a)
\]
\[
a_{uv} + K_{uv}^s \leq a_{uv}^+ + (1 - w_{uv})M \quad \text{for all } u \in U_+, s \in L, \delta_{uv} = 1 \quad (4.4b)
\]

For trains other than the current train $v$,
3.2. One train at a time at platforms, etc. For the current train \( v \) these constraints are as (3.5) in \( P^M \), but with \( x_{vst} \) replaced by \( w_{vst} \), thus

\[
\begin{align*}
    a_{us} + K_{wu}^u & \leq a_{u^+s} & \text{for all } u \in U_v, u \neq \bar{v}, s \in L \tag{4.4c'}
\end{align*}
\]

These constraints require that the headway between the current train \( v \) and the preceding train \( u \) is the same as the headway between \( u \) and the succeeding train \( u^+ \). For the remaining trains \( u(u \neq v) \), the constraints are

\[
\begin{align*}
    d_{us} + K_{wu}^{u^+} & \leq a_{u^+s} & \text{for all } u \in U_v, u \neq \bar{v}, s \in L \tag{4.4d'}
\end{align*}
\]

Note: The notation \( K_{wu}^{u^+} \) indicates that the preceding train \( u \) is traversing link \( s \) in the same direction as \( u^+ \), and \( a_{u^+s} \) the preceding train \( u \) is traversing link \( s \) in the opposite direction to \( u^+ \).

3.2. Two-way track constraints

To reduce the two-way track constraints (3.10a)-(3.10b) from program \( P^M \) to program \( P^o \) we proceed as follows (the steps are somewhat similar to the steps in reducing constraints (3.4a)-(3.5) above to (4.4a)-(4.5c)). Recall that \( U_i \) is the set of trains, (excluding the current train \( v \) already assigned to use link \( s \), in a fixed sequence order. Divide \( U_i \) into two sets,

\[
\begin{align*}
    U_i^+ &= \text{the set of trains in } U_i \text{ which traverse link } s \text{ in the "+" direction, and} \\
    U_i^- &= \text{the set of trains in } U_i \text{ which traverse link } s \text{ in the opposite or "-" direction.}
\end{align*}
\]

Hence to write headway constraints for the current train \( v \) with respect to other trains \( u \in U_i \), we have to consider four cases:

(i) the train \( u \) preceding train \( v \) traverses link \( s \) in the same direction as \( v \).
(ii) the train \( u \) preceding \( v \) traverses \( s \) in the opposite direction to \( v \).
(iii) the train \( v^+ \) succeeding \( v \) traverses link \( s \) in the same direction as \( v \).
(iv) the train \( v^+ \) succeeding \( v \) traverses link \( s \) in the opposite direction to \( v \).

Considering these four cases (i)-(iv) in order yields the following four subsets of headway constraints. First, suppose that the current train \( v \) is \( v \in V_i^+ \) and consider headways between train \( v \) and the preceding trains \( u \in U_i^- \).

(i) For trains \( u \in U_i^+ \) the ordinary "same-way" headway applies, as in (4.4a) and (4.4d), thus

\[
\begin{align*}
    a_{us} + K_{wu}^{u^+} & \leq a_{u^+s} + (1 - w_{u^+})M & \text{for all } u \in U_i^+, s \in L_i^+ & \tag{4.10a}
\end{align*}
\]

(ii) For trains \( u \in U_i^- \) constraints of the form in "One train at a time . . . " (4.5a) apply, thus

\[
\begin{align*}
    d_{us} + K_{wu}^{u^+} & \leq a_{u^+s} + (1 - w_{u^+})M & \text{for all } u \in U_i^-, s \in L_i^- & \tag{4.10c}
\end{align*}
\]

Now consider headways between train \( v \in V_i^- \) and succeeding trains \( u^+ \in U_i^+ \).

(iii) For trains \( u^+ \in U_i^+ \) the ordinary "same-way" headway constraints apply, as in (4.4b) and (4.4e), thus

\[
\begin{align*}
    a_{us} + K_{wu}^{u^+} & \leq a_{u^+s} + (1 - w_{u^+})M & \text{for all } u \in U_i^+, s \in L_i^- & \tag{4.10d}
\end{align*}
\]

Thus, the constraints (4.10a)-(4.10d) ensure that the headway between the current train \( v \) and the preceding train \( u \) is the same as the headway between \( u \) and the succeeding train \( u^+ \). For the remaining trains \( u(u \neq v) \), the constraints are

\[
\begin{align*}
    d_{us} + K_{wu}^{u^+} & \leq a_{u^+s} & \text{for all } u \in U_i, u \neq \bar{v}, s \in L \tag{4.4d'}
\end{align*}
\]
Train pathing model

\[ d_{vs} + K_{vs}^{d+} \leq d_{vs} + (1 - w_{vs})M \quad v \in V^+_s, \quad \text{for all } u \in U^+_s, s \in L^+ \]  
\[ (4.10\text{e}) \]

(iv) For trains \( u^+ \in U^-_s \), constraints of the form “One train at a time . . . ” (4.5b) apply, thus

\[ d_{vs} + K_{vs}^{d-} \leq d_{vs} + (1 - w_{vs})M \quad v \in V^-_s, \quad \text{for all } u \in U^-_s, s \in L^- \]  
\[ (4.10\text{f}) \]

If train \( v \) is \( v \in V^-_s \) (rather than \( v \in V^+_s \)) then the above discussion and four sets of constraints are all unchanged, except that \( V^+_s, U^+_s, \) and \( U^+_s \) are replaced with \( V^-_s, U^-_s, \) and \( U^-_s \), respectively throughout.

Now consider headway constraints between trains other than the current train \( v \). Again there are four possibilities. If both train \( u \) and its successor \( u^+ \) traverse link \( s \) in the same direction then the ordinary “same-way” link headway constraints apply, as in (4.4a) and (4.4f), thus

\[ a_{us} + K_{us}^{a+} \leq a_{us} + (1 - w_{us})M \quad v \in V^+_s, \quad \text{for all } u \in U^+_s, s \in L^+ \]  
\[ (4.10\text{g}) \]

\[ d_{us} + K_{us}^{d+} \leq d_{us} + (1 - w_{us})M \quad v \in V^+_s, \quad \text{for all } u \in U^+_s, s \in L^+ \]  
\[ (4.10\text{h}) \]

if \((u \in U^+_s, u^+ \in U^+_s)\) or \((u \in U^-_s, u^+ \in U^-_s)\).

On the other hand, if train \( u \) and its successor \( u^+ \) traverse link \( s \) in opposite directions then headway constraints similar to those in “One train at a time . . . ” (4.5c) apply, thus

\[ a_{us} + K_{us}^{a-} \leq a_{us} + (1 - w_{us})M \quad \text{for all } u \neq \overline{v}, s \in L^- \]  
\[ (4.10\text{i}) \]

if \((u \in U^+_s, u^+ \in U^-_s)\) or \((u \in U^-_s, u^+ \in U^+_s)\).

3.3. Optimization model \( P^M \)

The train pathing problem \( P^M \) for a single train \( v \) is now as already set out in Section 5.4 of Carey, plus the two-way track headway constraints, (4.10a)-(4.10i) from Section 3.2 above.

4. CONCLUSION

This paper extends the train pathing models and algorithms of Carey & Lockwood (1992) and Carey (1994), to handle two-way links (by extending Sections 2.2, 2.4, 5.2, and 5.4 of that paper). Introducing two-way track has not increased the number of constraints or variables, including 0–1 integer variables in the train pathing programs, i.e., the number of constraints and variables in these programs is the same for two-way track as for one-way track. For two-way track there are more sets of constraints [i.e., (4.10a)-(4.10i)] than for one way track [constraints (4.4a)-(4.5c)]. However, each of these sets applies to a smaller subset of cases (trains), obtained by dividing the set of trains \( V_s \) up into mutually exclusive subsets, e.g., \((v \in V^+_s, u \in U^+_s), (v \in V^-_s, u \in U^-_s), \) etc.

Furthermore, the solution strategies and algorithms set out in Carey and Lockwood (1994) and Carey (1994) still apply here, in particular, those in Sections 3 to 5 of the former and Section 4 of the latter. The reason for this is as follows. In program \( P^M \) the sequence order of all trains except the current train \( v \) is held fixed on each link. After each train on each link there is a potential time slot in which train \( v \) may be inserted, and the purpose of program \( P^M \) is to choose, for each link, the best slot in which to insert train \( v \). For one-way links train \( v \) has to slot in between existing trains so that its entry time (exit time) slots between their entry times (exit times). For two-way links (trains in both directions) train \( v \) again has to slot in between two trains, though there are now four alternative possibilities for these two trains, and hence for the headway constraints. (The two trains may be in the opposite direction to \( v \), or in the same direction as \( v \), or the first may be in the same direction and the second in the opposite direction, or vice versa.) Thus in principle the problem is the same for two-way links as for one-way links: that is choose one of the available time slots between the trains already assigned to each link.
With two-way links there is generally a very much longer interval between the time slots available to train $v$. If the preceding train is in the opposite direction to train $v$ then we have to wait for it to traverse the entire link and exit before train $v$ can enter. In contrast, with one-way links we merely have to wait until a few minutes departure headway have elapsed. The fact that there are much longer time intervals between time slots available for train $v$ tends to make solving the problem (i.e., $P^v$) much easier. The reason is as follows. In combinatorial problems, introducing additional (non-integer) potentially binding constraints tends to make solving the problem easier. In train planning and pathing there are normally bounds on the acceptable departure or arrival times for each train. If (as with two-way links) there are also large time intervals in which entering a link is not possible, this leaves relatively fewer time slots to choose from – fewer 0–1 integer variables.

For example, suppose it is required that train $v$ depart between 8:00 and 9:00 a.m., then on a one-way link with trains departing say about every 10 minutes, there are at most 6 or 7 time slots to choose from. In contrast, if trains between 8:00 and 9:00 a.m. are in the opposite direction to $v$ and take say 30 minutes to traverse the link, then there are at most 2 or 3 time slots between 8:00 and 9:00 to choose from. Assuming similar reductions in the number of time slots for other two-way links, the number of combinations to consider is likely to be greatly reduced when some of the links are two-way rather than one-way.

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