NONCONVEXITY OF THE DYNAMIC TRAFFIC ASSIGNMENT PROBLEM

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Abstract—We identify and discuss what appears to be a central difficulty for the future development of models of dynamic traffic flows on road networks. This difficulty is due to the fact that road traffic tends to behave in a first-in-first-out (FIFO) manner: that is, traffic which embarks on a road or other facility in period \( t \) exits from that facility ("on average") before traffic which enters in any later time periods. The FIFO requirement does not cause a problem in static traffic assignment, but we show that it yields a nonconvex constraint set in dynamic assignment, especially if there are multiple destinations or commodities. We consider various formulations, each of which yields a nonconvex optimization problem which is at present computationally tractable only for relatively small-scale examples. The above FIFO problem arises even if there is no congestion, and even if travel demands are fixed. Further the problem arises whether we are modeling a system optimum or a user equilibrium, and whether we use an optimization formulation or a complementarity or variational inequality formulation. We make some suggestions for dealing with, or avoiding, the problem and for further research.

A desired but elusive goal of research in dynamic traffic assignment is to develop well-behaved multiperiod network models analogous to well-known static or single period models. In particular, many authors have remarked on the need to extend the present limited scope dynamic assignment models so as to be able to handle multiple destinations and multiple traffic types. However, the development of such dynamic models runs into an underlying obstacle or problem which does not appear to have been explicitly identified in the literature. The obstacle consists of a basic nonconvexity in the dynamic traffic flow problem.

The purpose of this paper is to identify and draw attention to this nonconvexity problem. We suggest some ways in which one might attempt to deal with the problem, but we do not fully solve the problem or eliminate the difficulty. Instead, we identify the problem in the hope that this will motivate further research efforts to cope with, overcome or avoid it.

Throughout we consider the dynamic traffic assignment formulated as a multiperiod network model. However, for brevity and because it is not essential for our purpose here, we do not set out any explicit form for the constraints or objective function of the network model. What we have to say will apply to a variety of existing and possible formulations.

On a road network, traffic of different types which enters the same arc at approximately the same time will usually travel at the same speed. Individual vehicles do of course travel at different speeds and do pass each other. But in modeling aggregate flows we can normally assume that vehicles which are bound for different destinations, and which enter an arc at the same time, will take (approximately) the same time to traverse the arc. There are a variety of other ways in which this property of road traffic (and some other types of traffic) can be stated. One way is to say that traffic which enters an arc first will on average exit first, i.e. first-in-first-out, or FIFO. Another way is to say that traffic of different types (e.g. bound for different destinations) do not on average pass each other on an arc.

A relatively small number of optimization models have been developed to date to handle dynamic traffic assignment on networks (e.g. see Carey, 1986, 1987; Carey and Srinivasan, 1988; Friesz, Luque, Tobin, and Wie, 1988; Merchant, 1974; Merchant and Nemhauser, 1978a; Powell, 1991). A notable feature of all of these is that for road traffic they concentrate on a single destination or single commodity. However, as soon as we
attempt to move beyond these, to include multiple destinations or multiple commodities, we are immediately faced with how to ensure FIFO, or an analogous condition.

Our purpose in this paper is to discuss how this first-in-first-out (FIFO) or “no-passing” requirement can be captured or approximated in the constraints of mathematical programming models of network flows. We assume that, insofar as possible, the purpose is to add these additional FIFO constraints to existing mathematical programming models for multiperiod network flows.

We concentrate on considering approximating FIFO in continuous rather than seemingly “natural” discrete formulations for several reasons.

(a) Though discrete formulations of FIFO may seem more natural and exact, this is doubtful. To discretize, we treat the traffic entering each arc, or at each origin node, in each time period as a separate discrete chunk. But this chunking introduces inaccuracies which may multiply over successive arcs.

(b) If flows are treated as discrete, then this severely limits the size of the network model which can be solved.

(c) There has already been an enormous research investment over 20 years in developing and using continuous traffic assignment models. It seems prudent to see if these can be adapted to ensure FIFO.

1. FIFO IN STATIC ASSIGNMENT

Before turning to FIFO in time-varying (dynamic) traffic flows, it is worth briefly considering FIFO in static assignment models. Static traffic flow models do not explicitly consider the order of entry or exit from arcs, since they do not keep track of the time at which traffic enters or exits from an arc. However, in static assignment models, it has been usual to assume that the time taken to traverse any arc is the same for all traffic types and is independent of the origin or destination of the traffic. This is implicit in arc traversal time functions of the form \( t_c = f_c(v_c) \), where \( v_c \) is the flow rate of traffic type \( c \in C \) on arc \( j \). If the arc traversal time is the same for all traffic types, then traffic which enters first must exit first, i.e. FIFO holds.

However, different user classes (e.g. busses and cars) can in practice have significantly different average trip times (speeds) on the same link, and this has been recognised by various authors. To avoid formally introducing such issues here, we will assume in what follows that either there is only a single user class, or that user classes on a link do not have significantly different average trip times. Note that trip time is not the only component of trip cost, though it is usually the major component. Thus link trip costs can differ between two user classes on a link, even if their average trip times are the same.

2. FORMULATING FIFO BY USING DISCRETE OR INTEGER CONDITIONS

The FIFO (or no passing) requirement can be thought of as a sequencing or scheduling requirement, and as it is well known the latter can be formulated by using integer zero-one (0-1) variables. In the present (dynamic traffic flow) context FIFO can be formulated as follows. Let \( x_{nj} \) be the flow of traffic type \( c \) which enters arc \( j \) in period \( t = 1, \ldots, T \) and exits in period \( \tau > t \). FIFO requires that if \( x_{nj} > 0 \) then any flow which enters earlier than \( x_{nj} \) (i.e. before time \( t \)) cannot exit later than \( x_{nj} \) (i.e. after time \( \tau \)). More formally,

\[
(x_{nj} > 0) \Rightarrow \left( \sum_{c', \tau'} x_{c', \tau'} | t' < t, \tau' > \tau \right) = 0. \tag{1'}
\]

If there are multiple traffic types \( c \in C \), then FIFO requires that (1') holds for each traffic type \( c \) and also be extended to every pair of traffic types \( c, c' \in C \). Thus multicommodity FIFO can be stated as: for all \( t, j \) and \( c \),
Either-or restrictions such as (1) can be formulated as algebraic (LP) equations by using 0-1 variables [e.g. see Garfinkel and Nemhauser (1972), Chap. 1]. There are various alternative ways of writing (1) but they all result in a nonconvex constraint set. More formally:

**Proposition 1.** The above FIFO condition, (1') or (1) yields a nonconvex set.

**PROOF.** The following pair of flows satisfy (1): \( x_i^{*} > 0, x_i^{*} > 0, t^* < t', \tau^* > \tau' \), and \( \bar{x} = (x_i^{*}, \tau^* > 0, t^* < t', \tau^* > \tau') \). If the set of points satisfying (1) is convex, then any convex combination of points \( x \) and \( \bar{x} \) will satisfy (1). Consider the point \( x = (.5x_i^{*} + .5 \bar{x}) = (.5x_i^{*} > 0, .5x_i^{*} > 0) \). Clearly this does not satisfy (1), hence the proposition follows. \( \square \)

### 3. FORMULATING FIFO USING CONTINUOUS FLOW VARIABLES

In Section 1 we indicated how FIFO can be formulated algebraically by using discrete conditions or variables. Here we argue that for appropriately smoothly varying traffic flow conditions, we can use continuous variables and equations to represent or approximate the above discrete formulation of FIFO. Though this continuous formulation turns out to be nonconvex, it has the advantage of requiring no new variables and many fewer constraints than the discrete formulation.

#### 3.1. Single commodity case

Let \( x_{ij} \) denote the flow which enters arc \( j \) in period \( t \) and exits in period \( \tau \). Consider the set of flows embarking on arc \( j \) in period \( t \), i.e., \( x_{ij}, \tau = t, \ldots \). It follows immediately that:

(i) The aggregate inflow to arc \( j \) in period \( t \) is \( x_{ij} = \Sigma x_{ij}(\tau - t) \).

(ii) The aggregate time taken to traverse arc \( j \) (for flows \( \{x_{ij}, \tau = t, \ldots \} \) embarking on \( j \) in period \( t \)) is \( \Sigma x_{ij}(\tau - t) \). (To see this, recall that each flow \( x_{ij} \) takes \( (\tau - t) \) periods to traverse arc \( j \).

(iii) The average (mean) time taken to traverse arc \( j \) (for flows \( \{x_{ij}, \tau = t, \ldots \} \) embarking on \( j \) in period \( t \)) is obtained by dividing (ii) by (i), i.e.

\[
\bar{m}_{ij} = \frac{\sum_{\tau} x_{ij}(\tau - t)}{\sum_{\tau} x_{ij}}.
\]

Note that the equations (2) are nonlinear in \( x_{ij} \) and hence represent a nonconvex set.

(iv) For convenience we will use the term "average" or "typical" unit of flow to refer to the actual or hypothetical unit of flow which takes exactly the mean time \( \bar{m}_{ij} \) to traverse an arc. Thus, following (iii), the average unit of flow entering arc \( j \) at time \( t \) can be denoted by \( x_{ij} \), where \( \bar{t} = (t + \bar{m}_{ij}) \).

**Proposition 2.** For the single destination single commodity case, the constraints,

\[
\bar{m}_{ij} \leq 1 + \bar{m}_{i+1,j} \quad \text{for all } t,
\]

where \( \bar{m}_{ij} \) is defined in (iii) above, are

(a) necessary and sufficient to ensure FIFO for the average flows [see (iv)] \( \{x_{ij} \} \) on arc \( j \), and

(b) necessary (but not sufficient) to ensure FIFO for all the individual flow components \( \{x_{ij} \} \) on arc \( j \).
PROOF. (a) By definition of \( \bar{m}_{ij} \), the flows which enter arc \( j \) in any period \( t \), exit (on average) in any later period \( (t + \bar{m}_{ij}) \) and flows which enter in any later period \( (t + t') \) exit (on average) in period, \( (t + t') + \bar{m}_{ij} \). To ensure that the flow which entered first (in period \( t \)) exits first (i.e. FIFO), it is necessary and sufficient that \( (t + \bar{m}_{ij}) \leq ((t + t') + \bar{m}_{ij}) \). Cancelling \( t \) from each side gives \( \bar{m}_{ij} \leq t' + \bar{m}_{ij} \). However, the latter is equivalent to \((3)\), since it yields \((3)\) as a special case and can be obtained from \((3)\) by recursive substitution.

(b) The above condition is necessary for FIFO for the average flow (as defined above), but it is also necessary for FIFO for the individual time-space arc flows (without the averaging). The reason for this is of course that if the FIFO condition is violated by the average flows (as defined above), then FIFO must be violated for at least one of the individual components \( x_{\tau ij} \) of that average flow. \( \square \)

Condition \((3)\) is not sufficient to ensure FIFO for all components \( \{x_{\tau ij}, \tau = t + 1, \ldots, T\} \) of \( x_{ij} \). This is illustrated in Fig. 1. There the flow \( x_{ij} = x_{t+1,ij} \) which enters arc \( j \) in period \( t \) all exits in period \( (t + 3) \). The flow \( x_{t+1,ij} \) which enters in period \( t + 1 \) exits on average in say, period \( (t + 3.5) \), which satisfies FIFO on average. However, \( x_{t+1,ij} \) has two components, which exit in periods \( (t + 2) \) and \( (t + 4) \), respectively. The first of these components [entering at \((t + 1)\) and exiting at \((t + 2)\)] violates strict FIFO with respect to the flow which enters at time \( t \) and exits at time \((t + 3)\).

The violation of strict FIFO, as in Fig. 1, would be less likely to occur if the flows in Fig. 1 emanating from \((t + 1)\) had all arrived at the other end of the arc “bunched together” in neighboring time periods. For example, strict FIFO would hold if the flows from \((t + 1)\) arrived in say, periods \( (t + 3) \) and \( (t + 4) \), instead of in periods \( (t + 2) \) and \( (t + 4) \). More formally, strict FIFO is more likely to be satisfied if in any actual or realized set of flows (e.g. in an optimal solution) at most a set of neighboring members of \( \{x_{\tau ij}, \tau = t, \ldots, T\} \) are nonzero. This is a type of “special ordered set” (SOS) condition.

The above SOS condition could be imposed by introducing zero-one integer variables. However, instead of imposing any explicit SOS constraints we recommend first solving the network flow program without these, and then checking whether the solution happens to satisfy the SOS condition. In the case of road traffic flows varying relatively smoothly over time, we conjecture that the SOS condition will be satisfied (for most arcs and time periods) without having to be imposed explicitly. We also have some empirical evidence to support this, in Carey and Subrahmanian (1987).

3.2. Multicommodity case

The single user class, or single destination case considered above can be extended to multiple user classes and destinations. However, in the case of multiple user classes there is an additional complication, since different user classes may travel on average at different speeds, and may be subject to different degrees of interaction between user classes. To avoid discussing that problem here we can limit the discussion below to multiple destinations: The discussion applies also to multiple user classes if these classes travel on average at the same speed.

![Fig. 1. Flows satisfying average FIFO (3), but not strict FIFO (1).](image)
Items (i)–(iv) and most of the discussion in Section 3.1 extends to multiple destinations (commodities) by simply introducing a commodity subscript $c \in C$ on each of the variables. Thus (2), the mean time taken to traverse an arc, becomes

$$m_{ijc} = \left[ \sum_{\tau=1}^{T} x_{ijc}(\tau - t) \right] / \sum_{\tau=1}^{T} x_{ijc}.$$  \hspace{1cm} (2')

The average FIFO condition (3) becomes,

$$m_{ijc} \leq 1 + m_{i+1,jc} \quad \text{for all } t, c \in C.$$  \hspace{1cm} (3')

However, (3') is no longer sufficient in part (a) of Proposition 2, since it ensures FIFO only within each commodity type, but not across commodities. There are various ways in which the latter can be ensured.

One way of ensuring FIFO across commodities is to rewrite (3') for all pairs of commodities, thus,

$$m_{ijc} \leq 1 + m_{i+1,jc'} \quad \text{for all } t, j, c, c' \in C.$$  \hspace{1cm} (3")

However, this requires a very large number of constraints, namely $|C|^2$ constraints, where $|C|$ is the number of elements in $C$.

A second way of ensuring intercommodity FIFO is to ensure that the arc traversal times are the same for all commodities. Thus introduce constraints,

$$m_{ij} = m_{ijc} \quad \text{for all } j, t \text{ and } c,$$  \hspace{1cm} (4)

so that $m_{ij}$ becomes the average arc traversal time for each commodity. Then introduce (3), i.e. without a commodity subscript, to ensure that these $m_{ij}$'s satisfy FIFO.

A third way to ensure FIFO across commodities is as follows. Consider the traffic of various types ($c, c' \in C$) which enters arc $j$ at time $t$ and exits spread over some subset of the periods $\tau > t$. FIFO requires that the distribution of this (exiting) traffic over periods $\tau > t$ is the same for all commodity types $c \in C$. More formally,

$$\left( \frac{x_{ijc}}{\sum_{\tau > t} x_{ijc}'} \right) = \left( \frac{x_{ijc'}}{\sum_{\tau > t} x_{ijc}''} \right)$$  \hspace{1cm} (5)

for all $t > t$ and $c, c' \in C$. These are nonlinear equations and hence represent a nonconvex constraint set.

The preceding three paragraphs enable us to extend Proposition 2 to multiple commodities, thus:

**Proposition 3** For each arc $j$ the conditions (2'), (3") or (2'), (3), (4) or (5) are,

(a) necessary and sufficient to ensure intra- and intercommodity FIFO for the average flows $\{x_{ij},\}$ for all $t$ and $c$, where $\bar{r} = m_{ij}$, and

(b) necessary (but not sufficient) to ensure intra- and intercommodity FIFO for all the individual flow components $\{x_{ijc},\}$ for all $t, \tau > t$, and $c, c'$.

The above formulations [(2)–(5)] could be used to approximate or represent FIFO when the network flows vary sufficiently smoothly over time for the approximation to be acceptable. Network flows are more likely to vary smoothly over time for road traffic flows than for say, production or manufacturing flows. It may not be possible to tell in advance whether the flows in the solution of a network model will vary smoothly over time. We therefore recommend first solving the network flow model without introducing any explicit FIFO restriction, to see how or where FIFO is violated. This also enables one
to see whether the flows obtained are varying sufficiently smoothly over time for the continuous formulations [(2)-(5)] of the FIFO conditions to be sufficiently accurate for the context or problem in hand.

4. FORMULATING FIFO BY USING CONTINUOUS STOCK AND FLOW VARIABLES

In Section 2 we indicated how FIFO can be formulated by using discrete conditions or variables. In Section 3 we showed that, for appropriate traffic conditions, we can use continuous variables to approximate the above discrete formulations. Here we will consider FIFO in the context of a different continuous model.

One of the efforts to develop a dynamic traffic assignment model for congested networks is that in Merchant and Nemhauser (1978), Carey (1987), Carey and Srinivasan (1988). We will not set out this (MNC) model here: we note only that it contains inflow variables, outflow variables, and arc volume variables, for each arc \( j \) for each period \( t \). The volume on an arc is denoted by \( x_j \) and the outflow from an arc in period \( t \) by \( y_j \). We can't introduce FIFO here as in Section 3 above, since in the MNC model, even though we know the flow exiting from arc \( j \) in period \( t \), we do not know in what period(s) this flow entered the arc. Instead, recall an assumption which is already present in that model, homogeneity: i.e. on arc \( j \) the mixture of traffic types is assumed to be distributed approximately uniformly along the arc, so that the outflow at time \( t \) is a function only of the total volume \( x_j \) on the arc. Given this assumption, a natural form of FIFO condition is that the mixture of traffic types exiting from the arc (i.e. \( \{y_j\} \)) should be the same as the mixture of traffic types \( \{x_j\} \) actually on on the arc. More formally,

\[
\frac{x_{jc}}{x_j} = \frac{y_{jc}}{y_j} \quad \text{for all } c \in C. \tag{6}
\]

An equivalent way of writing (6) is, \( (x_{jc}/x_j) = (y_{jc}/y_j) \) for all \( c \in C, c' \in C \).

Equation (6) is nonlinear, and hence represent nonconvex constraint sets. Even if we multiply through (6) to give \( y_{jc} = (x_{jc}y_j) \), this of course still represents a nonconvex set, and even if we change the "=" in (6) to "≥" or "≤" the constraint set is still nonconvex. However, though (6) is nonconvex it may be preferable to the discrete (nonconvex) formulation (1), since it requires many fewer constraints and allows us to use continuous nonlinear programming algorithms.

If there are no congestion effects present, then a dynamic multicommodity, or multidestination, model of traffic flows with FIFO can now be obtained by adding (6) to the usual formulation of the linear, least-cost, multicommodity network flow model with multiple time periods. If congestion is present then a model of congested multicommodity (or multiple destination) network flows over time can be obtained by adding the FIFO constraints (6) to a multicommodity version of the MNC model.

Intuitively it seems that the homogeneity assumption stated above is a better approximation to reality for traffic flows on each arc \( j \), (a) the more smoothly the traffic inflow and outflow from arc \( j \) varies (grows and/or declines) over time. (b) the shorter is arc \( j \), and (c) the shorter are the time periods \( t \).

5. CONCLUDING REMARKS

The discussion and results in this paper are independent of whether the traffic flows are congested or uncongested, or whether the origin-destination travel demands are constant or variable (price elastic), or whether we are modelling a "system optimum" or a "user equilibrium".

In referring to convex and nonconvex constraint sets throughout, we have implicitly been concerned with an optimization formulation. However, traffic network models have also been formulated as complementarity and variational inequality problems (Ashtiani and Magnanti, 1981; Dafermos, 1980; Florian, 1984; Magnanti, 1984). Introducing FIFO
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constraints [such as (1)-(6)] into these formulations causes at least as great a computational problem as it does for an optimization formulation. Indeed, one of the most popular and effective ways of solving complementarity and variational inequality problems is to iteratively solve a sequence of optimization problems, each of which is a local approximation to the given complementarity or variational inequality problem.

Throughout, we have assumed, as usual, an “arc-node” formulation for network flows instead of the alternative “arc-path” formulation. However, essentially the same results and discussion hold for an arc-path formulation.

How serious are the computational difficulties imposed by FIFO likely to be in practice in solving multiperiod, multicommodity, traffic network problems? The answer is largely empirical. An obvious strategy is to first solve the given network model without imposing any explicit FIFO constraints. Then check how many FIFO violations occur, and how serious they are. For the case of traffic flows which vary relatively smoothly over time, with no sudden sharp rises and falls in traffic flows, we conjecture that FIFO conditions will tend to be automatically satisfied, or at least approximately satisfied, without being explicitly imposed. We have some empirical evidence for this in Carey and Subrahmanian (1987). The degree of approximation to FIFO which is acceptable depends of course on the decision maker and the problem in hand.

We have not in this paper discussed any algorithms for solving the zero-one integer or nonlinear nonconvex programming problems which we formulated. Instead we focused on identifying and characterizing the problem. We hope that this will concentrate additional research on the problem so that it can be solved, or circumvented or at least better understood.

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REFERENCES